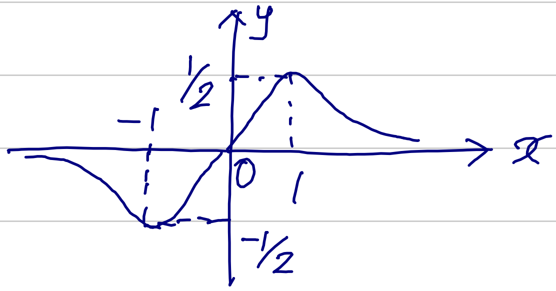
Solutions

MAS 140/151/152/156/161 2016–17

A1 $y = \dfrac{x}{x^2+1}$ is odd.

$$y' = \frac{x^2+1 - x\cdot 2x}{x^2+1} = \frac{1-x^2}{(x^2+1)^2}$$

At $x=1$, $y = 1/2$ (maximum) M1A1



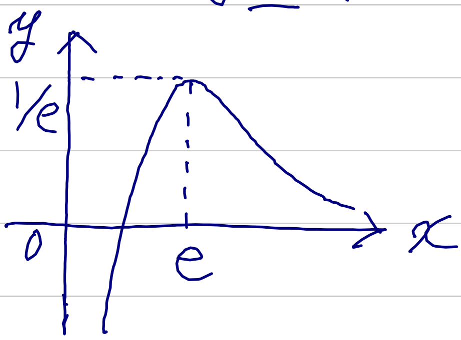
A1

A2 $y = \dfrac{\ln x}{x}$ is defined on $x>0$ A1

$$y' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

Maximum at $x=e$, $y = 1/e$

Hence $y \le 1/e$ M1A1



(graph not necessary)

A3 $f = e^x \cdot \sin y$

$$\Rightarrow \begin{cases} f_x = e^x \sin y \\ f_y = e^x \cos y \\ f_{xy} = e^x \cos y \end{cases}$$

A3

A4 $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2}$

$= \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$ M1 A2

alternatively

$\frac{x-\sin x}{x^3} = \frac{x-(x-\frac{1}{3!}x^3+O(x^5))}{x^3}$

$= \frac{1}{3!} + O(x^2) \to \frac{1}{6}$ (as $x\to 0$)

A5 Let $z = x+iy$.

$|x+i(y-1)| = |x+i(y+1)|$

$\Leftrightarrow x^2+(y-1)^2 = x^2+(y+1)^2$ M1 A1

$\Leftrightarrow -2y = 2y$

$\Leftrightarrow y = 0$

Hence $z = {}^{\forall}x \in \mathbb{R}$ A1

Alternatively, by inspection, the locus of points equidistant from i and $-i$ is the real axis.

A6 If $a = (1,1,1)$ and $b = (1,-2\lambda,1)$ are $\perp$

$a\cdot b = 1-2\lambda+1 = 0 \Rightarrow \lambda = 1$ M1 A1

If they are $/\!/$, $-2\lambda = 1 \Rightarrow \lambda = \frac{-1}{2}$ A1

A7 $\int_0^1 xe^{-x}dx = [-e^{-x}x]_0^1 + \int_0^1 e^{-x}\cdot 1\,dx$ M1 A1

(from warm-up) $\qquad \int_0^1 e^{-x}\cdot 1\,dx = [-e^{-x}]_0^1$

$= -e^{-1} - e^{-1} + 1$

$= 1-2e^{-1}$ A1

A8 $\int x \ln x\,dx = \frac{x^2}{2}\ln x - \int \frac{x^{\cancel{2}}}{2}\frac{1}{\cancel{x}}\,dx$ M1A1

(from warm-up) $= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$ A1

A9 Because $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ represents rotation through an angle of θ,

$$A^{-1} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ M1A1

(Alternatively, direct evaluation is also OK.)

Thus, $A^{-1} = A^T$, hence $AA^T = A^TA = I$ A1

A10 The condition for a unique solution is

$\det(A) = 3 - 2\lambda \neq 0$

that is, $\lambda \neq 3/2$. M1A1

The condition for infinitely many solutions is

$\det(A) = 3 - 2\lambda = 0$ A1

A11

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

$\tan^{-1}y = \tan^{-1}x + C'$ M1A1

Write $C' = \tan^{-1}C$ for $\exists C$

$y = \tan(\tan^{-1}x + \tan^{-1}C) = \frac{x+C}{1-Cx}$ // A1

($y = \tan(\tan^{-1}x + C')$ is also OK!)

A12 $\quad y' + y = x \quad ①$

Noting $\quad (ye^x)' = e^x(y'+y)$

$① \Leftrightarrow e^{-x}(ye^x)' = x$ $\quad$ M1A1

$\Leftrightarrow (ye^x)' = xe^x$

$\Leftrightarrow ye^x = \int xe^x\,dx$

$= xe^x - \int e^x\,dx$

$= xe^x - e^x + C$

Thus $\quad y = x - 1 + Ce^{-x}$ $\quad$ A1

B1 (i) $\quad g(y) = \sin^{-1}y + y\sqrt{1-y^2}$

(Similar to prob. sheet)

$$g'(y) = \frac{1}{\sqrt{1-y^2}} + \sqrt{1-y^2} + \frac{-y^2}{\sqrt{1-y^2}}$$

$= 2\sqrt{1-y^2}$ $\quad$ M1A2

(ii) $\quad f(x) = g\left(\frac{x}{a}\right)$

$$\frac{df}{dx} = g'\left(\frac{x}{a}\right)\cdot\frac{1}{a} \quad \text{(chain rule)}$$

Taking $y = \frac{x}{a}$,

$$\frac{df}{dx} = 2\sqrt{1-\left(\frac{x}{a}\right)^2}\ \frac{1}{a}$$ M1A2

$$\Leftrightarrow \frac{2}{a}\int\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx = f(x) + C'$$

$$= \sin^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2} + C'$$

By $\times\frac{a^2}{2}$;

$$\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

M1A1

B2 $z=\sqrt{3}+i$

$$\begin{cases} r=\sqrt{3+1}=2 \\ \tan\theta = 1/\sqrt{3} \Rightarrow \theta = \pi/6 \end{cases}$$

Hence polar form of z is, for $p\in\mathbb{Z}$

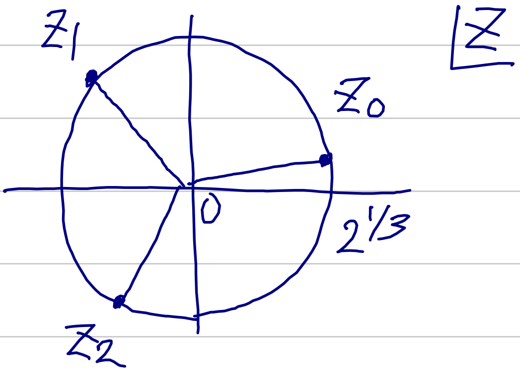
$$z=2\left[\cos\left(\frac{\pi}{6}+2\pi p\right)+i\sin\left(\frac{\pi}{6}+2\pi p\right)\right]$$ M1A1

Thus

$$z^{1/3}=2^{1/3}\left[\cos\left(\frac{\pi}{18}+\frac{2\pi p}{3}\right)+i\sin\left(\frac{\pi}{18}+\frac{2\pi p}{3}\right)\right]$$ M1A1

Choosing $p=0,1,2$, we find

$$\begin{cases} z_0 = 2^{1/3}\left[\cos\frac{\pi}{18}+i\sin\frac{\pi}{18}\right] \\ z_1 = 2^{1/3}\left[\cos\frac{13\pi}{18}+i\sin\frac{13\pi}{18}\right] \\ z_2 = 2^{1/3}\left[\cos\frac{25\pi}{18}+i\sin\frac{25\pi}{18}\right] \end{cases}$$ A3



A1

B3 $\mathbf{r}=(t^2+t,\ t^3-t,\ t^3-t^2)$

(i) $\mathbf{v}=\dot{\mathbf{r}}=(2t+1,\ 3t^2-1,\ 3t^2-2t)$,

$\mathbf{a}=\ddot{\mathbf{r}}=(2,\ 6t,\ 6t-2)$

At $t=1$ $\quad \mathbf{v}=(3,2,1)$

$\mathbf{a}=(2,6,4)$ M2A2

(ii) $\mathbf{t}=\frac{\mathbf{v}}{|\mathbf{v}|}=\frac{(3,2,1)}{\sqrt{14}}$ M1A1

$\therefore a_t=\mathbf{a}\cdot\mathbf{t}=\frac{(3,2,1)}{\sqrt{14}}\cdot(2,6,4)=\frac{22}{\sqrt{14}}=\frac{11}{7}\sqrt{14}$ M1A1

B4 (i) Characteristic poly. is

$$\lambda^2 - (a+1)\lambda + a = 0$$

$$(\lambda - 1)(\lambda - a) = 0 \qquad \therefore \ \lambda = 1, a$$

When $a \neq 1$, there are two indep. sols. e^x, e^{ax}.

$$\therefore \ y = c_1 e^x + c_2 e^{ax}$$ M1A2

(ii) When $a = 1$, characteristic poly is

$$(\lambda - 1)^2 = 0, \quad \lambda = 1 \text{ only}.$$

i.e. repeated root.

$$\therefore \ y = c_3 e^x + c_4 x e^x$$ M1A2

(iii) $\dfrac{e^{ax} - e^x}{a-1}$ is a linear combination of sols. obtained in (i), which solves the ODE. Now,

$$\lim_{a \to 1} \frac{e^{ax} - e^x}{a-1} = \lim_{a \to 1} \frac{x e^{ax}}{1} = x e^x$$

In the limit $a \to 1$, this sol. converges to the 2nd sol of (ii). M1A1

B5

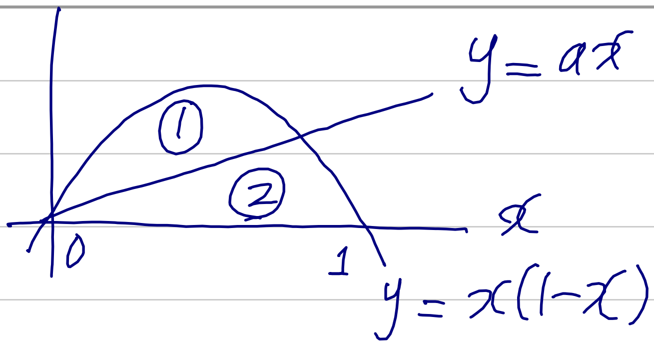
(i) $$\int_0^\alpha x(\alpha - x)\,dx = \int_0^\alpha (\alpha x - x^2)\,dx$$

$$= \left[\frac{\alpha}{2}x^2 - \frac{x^3}{3}\right]_0^\alpha$$

$$= \frac{\alpha^3}{6}$$ M1A1

(ii)

(unseen)



Intersection is given by $ax = x(1-x)$

If $x \neq 0$, $a = 1-x$ $\therefore x = 1-a$. M2A1

By (i), $\begin{cases} ① + ② = 1/6 \\ ① = \int_0^{1-a} \{x(1-x) - ax\} dx = \int_0^{1-a} x\{(1-a) - x\} dx = \frac{(1-a)^3}{6} \end{cases}$

The condition says ① = ② $\therefore 2 \cdot \frac{(1-a)^3}{6} = \frac{1}{6}$

$\therefore (1-a)^3 = \frac{1}{2}$

$\therefore a = 1 - \frac{1}{\sqrt[3]{2}}$ M2A1

B6 Char. poly. is

$$\begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & 4-\lambda & 1 \\ 2 & -4 & -\lambda \end{vmatrix} = 0$$

$$\frac{(1-a)^3}{6}$$

$\Leftrightarrow \begin{vmatrix} -1 & 4-\lambda \\ 2 & -4 \end{vmatrix} - \begin{vmatrix} 1-\lambda & 2 \\ 2 & -4 \end{vmatrix} - \lambda \begin{vmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{vmatrix} = 0$

or $4 - 2(4-\lambda) + 4(1-\lambda) + 4 - \lambda\{(1-\lambda)(4-\lambda) + 2\} = 0$

$\cancel{4} - \cancel{8} + 2\lambda + \cancel{4} - 4\lambda + 4 - \lambda(\lambda^2 - 5\lambda + 6) = 0$

or $-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

or $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0$

$(\lambda - 1)(\lambda - 2)^2 = 0$

$\therefore \lambda = 1, 2$ (repeated) M2A2

• $\lambda=1$, $\begin{pmatrix} 0 & 2 & 1 \\ -1 & 3 & 1 \\ 2 & -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$ $\therefore \begin{cases} 2y+z=0 \\ -x+3y+z=0 \\ 2x-4y-z=0 \end{cases}$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

M1A1

• $\lambda=2$, $\begin{pmatrix} -1 & 2 & 1 \\ -1 & 2 & 1 \\ 2 & -4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=0$

$\therefore x-2y-z=0$ pick any two linearly indep. vectors on the plain

e.g. $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$

M1A1

(due to Euler)

B7 By Hint (1)

$$I=\frac{1}{2}\int_0^{\pi/2}\ln\frac{\sin 2x}{2}\,dx \quad (y=2x)$$

$$=\frac{1}{2}\int_0^{\pi}\ln\left(\frac{\sin y}{2}\right)\cdot\frac{dy}{2}$$

$$=\frac{1}{4}\left(\underbrace{\int_0^{\pi}\ln\sin y\,dy}_{\equiv J}-\underbrace{\int_0^{\pi}\ln 2\,dy}_{=\pi\ln 2}\right)$$

M2A2

But

$$J=\underbrace{\int_0^{\pi/2}\ln\sin y\,dy}_{=I}+\int_{\pi/2}^{\pi}\ln\sin y\,dy \quad (z=y-\pi/2)$$

$$\int_{\pi/2}^{\pi}\ln\sin y\,dy=\int_0^{\pi/2}\ln\cos z\,dz=I$$

$\therefore 4I=2I-\pi\ln 2$ $\therefore I=-\frac{\pi}{2}\ln 2$

M2A2

Alternatively, by Hint (2) we start from first principles.

$$\int_0^{\pi} \ln \sin x \, dx = \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n-1} \ln \sin \frac{k\pi}{n} \qquad \text{M1A1}$$

$$= \lim_{n\to\infty} \frac{\pi}{n} \ln \prod_{k=1}^{n-1} \sin \frac{k\pi}{n} \qquad \text{M1A1}$$

$$= \lim_{n\to\infty} \frac{\pi}{n} \ln \frac{n}{2^{n-1}}$$

$$= \lim_{n\to\infty} \pi \left(\frac{\ln n}{n} - \frac{n-1}{n} \ln 2 \right) \qquad \text{M1A1}$$

$$= -\pi \ln 2$$

$$\therefore I = \frac{1}{2} \int_0^{\pi} \ln \sin x \, dx = -\frac{\pi}{2} \ln 2 \qquad \text{M1A1}$$

B8

$$\begin{array}{c} \times 2 \\ \\ \times 5 \end{array} \begin{pmatrix} 1 & 2 & 7 & 3 \\ -2 & 5 & 4 & 3 \\ -5 & 6 & -3 & 1 \end{pmatrix}$$

$$\rightarrow \begin{array}{c} \\ 1/9 \\ 1/16 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 9 & 18 & 9 \\ 0 & 16 & 32 & 16 \end{array}\right)$$

$$\rightarrow \begin{array}{c} \\ \\ \times(-1) \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \text{M2A2}$$

Hence,

$$\begin{cases} x + 2y + 7z = 3 \\ \qquad y + 2z = 1. \end{cases} \qquad \text{M1A1}$$

Put $z = \lambda$, then

$y = 1 - 2\lambda$

$x = 1 - 3\lambda$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-3\lambda \\ 1-2\lambda \\ \lambda \end{pmatrix} \qquad \text{M1A1}$$

$$\underbrace{\left\{ = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \right\}}_{\text{optional}}$$