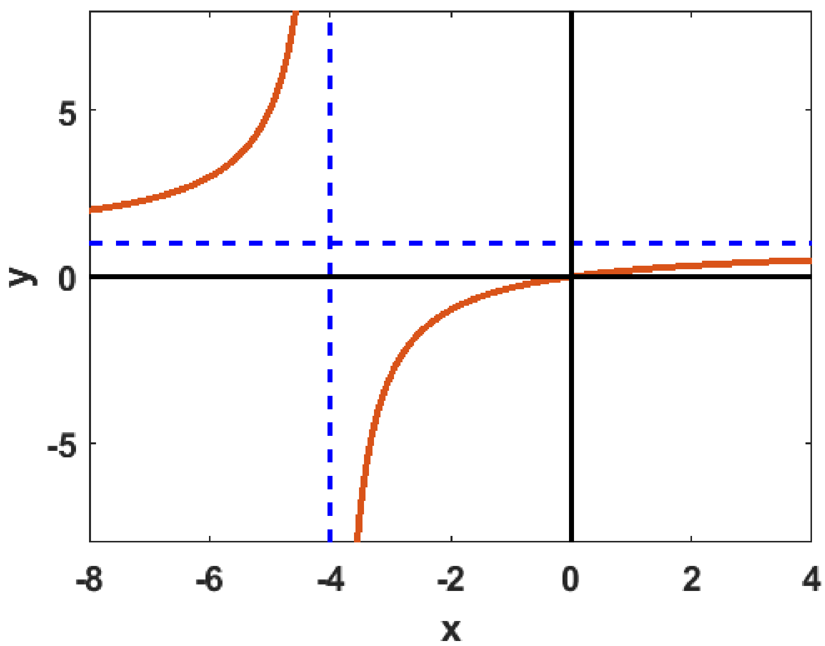
Solutions to MAS140_151_152_156 2015-16

(A1)

$$\frac{x}{x+4}$$



Graph of $y = \frac{x}{x+4}$, dashed lines are asymptotes **(1 mark)**, thick curve the function `(1 mark)`. Must show these with x and y values clearly marked **(1 mark)**.

(3)

(A2) $y = e^{2x} - 1 \Rightarrow ln(y+1) = 2x \Rightarrow x = \frac{1}{2}ln(y+1)$. **(1M)**

Hence $f^{-1}(x) = \frac{1}{2}ln(x+1)$. **(1A)**

Domain $x > -1$, range $x \in \mathbb{R}$. **(1A)**

(3)

(A3) $f(x, y) = 4x^2\sqrt{y} + 5cos(xy)$.

$\frac{\partial f}{\partial x} = 8x\sqrt{y} - 5y\sin(xy)$

$\frac{\partial f}{\partial y} = 4x^2\frac{1}{2\sqrt{y}} - 5x\sin(xy) = \frac{2x^2}{\sqrt{y}} - 5x\sin(xy)$. **(1M, 2A)**

(3)

(A4) At $x = 0$, $x \tanh x = 0$ and $\sin 2x = 0$, hence need to apply l'Hôpital's Rule.
$\lim_{x\to 0}\left(\frac{x\tanh x}{\sin 2x}\right) = \lim_{x\to 0}\left(\frac{\tanh x + x\,\mathrm{sech}^2\, x}{2\cos 2x}\right) = 0$. **(2M, 1A)**

(3)

(A5) Let $z = a + ib$. Since Re(z) =Im(z) we have $z = a + ia = a(1+i)$. $|z - 1 - i| = 1 \Rightarrow (a-1)^2 + (a-1)^2 = 2(a-1)^2 = 1 \Rightarrow a - 1 = \pm\frac{1}{\sqrt{2}}$.
Hence $z = (1 \pm \frac{1}{\sqrt{2}})(1+i)$. **(1M, 2A)**

(3)

(A6) $\mathbf{a} = (7, -2, -5)$, $\mathbf{b} = (5, 1, 3)$ $\mathbf{a}\cdot\mathbf{b} = 7\times 5 + -2\times 1 + -5\times 3 = 35 - 2 - 15 = 18$. **(1A)**

Vector perpendicular to $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -2 & -5 \\ 5 & 1 & 3 \end{vmatrix}$
$= \mathbf{i}(-2\times 3 - (-5)\times 1) - \mathbf{j}(7\times 3 - (-5\times 5) + \mathbf{k}(7\times 1 - (-2)\times 5)$
$= -\mathbf{i} - 46\mathbf{j} + 17\mathbf{k}$. **(1M, 1A)**

(3)

(A7)

$$\begin{aligned}\int_0^{\pi}(x+1)\sin\frac{x}{2}\,dx &= \left[(x+1)(-2\cos\frac{x}{2})\right]_0^{\pi} - \int_0^{\pi}(-2\cos\frac{x}{2})\,dx \\ &= 2 + 2\left[2\sin\frac{x}{2}\right]_0^{\pi} = 2 + 4 = 6.\end{aligned}$$

(2M, 1A)

(3)

(A8) $\int x \cos^2 x^2 \, dx$.

First make the substitution $u = x^2$, $\dfrac{du}{dx} = 2x$

$\displaystyle\int x(\cos x^2)^2 \, dx = \int \frac{1}{2} \cos^2 u \, du.$

Now use double angle formula $\cos 2u = 2\cos^2 u - 1$.

$$\begin{aligned}\int \frac{1}{2}\cos^2 u \, du &= \int \frac{1}{4}(\cos 2u + 1)\, du \\ &= \frac{1}{4}\left(\frac{1}{2}\sin 2u + u\right) \\ &= \frac{1}{8}\sin 2x^2 + \frac{1}{4}x^2.\end{aligned}$$

(2M (any), 1A)

(3)

(A9) $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$ and $B^T = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$. **(1A)**

$(AB) = \begin{bmatrix} 1 \times 2 + 1 \times (-1) & 1 \times 1 + 1 \times 3 \\ 3 \times (-1) & 3 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 9 \end{bmatrix}.$

$(AB)^T = \begin{bmatrix} 1 & -3 \\ 4 & 9 \end{bmatrix}$ **(1A)**

$B^T A^T = \begin{bmatrix} 2 \times 1 - 1 \times 1 & -1 \times 3 \\ 1 \times 1 + 3 \times 1 & 3 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 9 \end{bmatrix} = (AB)^T.$ **(1A)**

(3)

(A10) Rewrite the equation as

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{e^x}{x^3}.$$

Integrating factor is $= e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = e^{\ln x^3} = x^3$. **(1M)**

Hence we have

$$\begin{aligned} \frac{d(yx^3)}{dx} &= e^x \\ yx^3 &= e^x + c \\ y &= \frac{e^x}{x^3} + \frac{c}{x^3}. \end{aligned}$$

(1M, 1A)

(3)

(A11) Write the equations in matrix form $A\mathbf{x} = \mathbf{b}$ hence $\mathbf{x} = A^{-1}\mathbf{b}$.

$$\begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ -8 \end{pmatrix}.$$

$|A| = -6 - 5 = -10$ and $A^{-1} = \frac{-1}{10}\begin{pmatrix} -3 & -4 \\ -1 & 2 \end{pmatrix}$. **(2M)**

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{10}\begin{pmatrix} -3 & -4 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 14 \\ -8 \end{pmatrix} = \frac{-1}{10}\begin{pmatrix} -3\times 14 - 4\times(-8) \\ -14 - 2\times 8 \end{pmatrix} = \frac{-1}{10}\begin{pmatrix} -10 \\ -30 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(1A)

(3)

(A12) Looking for solutions of the form e^{mx}.

Hence we have $m^2 + 2m + 14 = 0 \Rightarrow m = \frac{-2\pm\sqrt{4-16}}{2} = -1 \pm \sqrt{3}i$. **(1M)**

So the general solution is of the form

$$y = e^{-x}(A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x)).$$

When $x = 0$, $y = A$, hence we must have $A = 0$ and the general solution is

$$y = Be^{-x}\sin(\sqrt{3}x).$$

(1M, 1A)

(3)

(B1)

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 - 1$$

For stationary points

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = 0$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y = 0$$

(2M)

Adding we have $x^3 + y^3 = 0 \Rightarrow x = -y$ and substituting into the first equation (or otherwise) $x^3 - 2x = 0 \Rightarrow x^2 = 0,\ 2$.

Hence the stationary points are $(0,0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$. **(3A)**

We require the second derivatives evaluated at the stationary points:

$$\begin{aligned} f_{xx} &= 12x^2 - 4 = 20 \ \text{ at both non-zero points} \\ f_{yy} &= 12y^2 - 4 = 20 \ \text{ at both non-zero points} \\ f_{xy} &= 4 \quad \text{everywhere} \end{aligned}$$

(1M)

Hence $\Delta = f_{xx}f_{yy} - (f_{xy})^2 = 20 \times 20 - 16 > 0$ and both points are minima. **(1M, 1A)**

(8)

(B2)

$$
\begin{aligned}
y &= \ln(1+x^2) \ = 0 \ @ \ x = 0 \\
y' &= \frac{1}{1+x^2}(2x) \ = 0 \ @ \ x = 0 \\
y'' &= \frac{2}{1+x^2} - \frac{2x2x}{(1+x^2)^2} \\
&= \frac{2(1+x^2-2x^2)}{(1+x^2)^2} \\
&= \frac{2(1-x^2)}{(1+x^2)^2} \ = 2 \ @ \ x = 0 \\
y''' &= \frac{-4x}{(1+x^2)^2} - \frac{4(1-x^2)2x}{(1+x^2)^3} \\
&= \frac{4x(x^2-3)}{(1+x^2)^3} \ = 0 \ @ \ x = 0 \\
y'''' &= 4\frac{(x^2-3)+x(2x)}{(1+x^2)^3} - 12\frac{x(x^2-3)2x}{(1+x^2)^4} \\
&= 12\frac{6x^2-x^4-1}{(1+x^2)^4} \ = -12 \ @ \ x = 0
\end{aligned}
$$

(2M, 2a (0.5 each))

Note that in the above it is not necessary to 'simplify' the expressions to determine the value of the derivatives at $x = 0$.

Hence the first two non-zero terms of the Maclaurin series are

$$y = \frac{2x^2}{2!} - \frac{12x^4}{4!} = x^2 - \frac{x^4}{2}.$$

(1M, 1A)

From the Formula Sheet $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$.

Replacing x with x^2 we have

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4}$$

and the first two terms are the same. **1M, 1A**

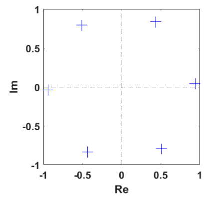
(8)

(B3) $z_1 = 1+i$, $|z_1| = \sqrt{1+1} = \sqrt{2}$, $\arg(z_1) = \frac{\pi}{4}$. Hence $z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$. **(1M, 1A)**

$z_2 = \sqrt{3}+i$, $|z_2| = \sqrt{(3+1)} = 2$, $\arg(z_2) = \frac{\pi}{6}$. Hence $z_2 = 2e^{i\frac{\pi}{6}}$. **(1M, 1A)**

Hence $z^6 = \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}e^{i(\frac{\pi}{4}-\frac{\pi}{6})} = \frac{1}{\sqrt{2}}e^{i(\frac{\pi}{12}+2\pi n)}$.

Hence $z = \frac{1}{2^{\frac{1}{12}}}e^{i(\frac{\pi}{72}+\frac{n\pi}{3})}$ for $n = 1, \cdots, 6$. **(2M, 1A)** The six solutions



are marked with + in the figure. The plot does not have to be exact but should make it clear that they lie on a circle (same modulus) separated by $\frac{\pi}{3}$ in argument with none on the Re or Im axes. **(1A)**

(8)

(B4) (a)

$$\begin{aligned} \mathbf{r}(t) &= (2t^2, t^2-4t, 3t-5) \\ \frac{d\mathbf{r}}{dt} &= (4t, 2t-4, 3) \\ \frac{d^2\mathbf{r}}{dt^2} &= (4, 2, 0) \end{aligned}$$

(2A)

(b) unit vector, at $t = 1$, $\hat{r}(t) = \frac{(2,-3,-2)}{\sqrt{4+9+4}} = \frac{(2,-3,-2)}{\sqrt{17}}$. **(1M, 1A)**

Hence components of velocity and acceleration in this direction are:

$$\begin{aligned} \frac{d\mathbf{r}}{dt}\cdot\hat{r}(t) &= (4,-2,3)\cdot\frac{(2,-3,-2)}{\sqrt{17}} = \frac{8}{\sqrt{17}} \\ \frac{d^2\mathbf{r}}{dt^2}\cdot\hat{r}(t) &= (4,2,0)\cdot\frac{(2,-3,-2)}{\sqrt{17}} = \frac{2}{\sqrt{17}} = \frac{1}{4}\frac{8}{\sqrt{17}} \end{aligned}$$

(2M, 2A)

(8)

(B5)

$$\int_1^2 \frac{2t^2+3t+1}{t^3+t}\,dt = \int_1^2 \frac{2t^2+3t+1}{t(t^2+1)}\,dt$$

Need to use partial fractions.

$$\begin{aligned}\frac{2t^2+3t+1}{t(t^2+1)} &= \frac{A}{t}+\frac{Bt+C}{1+t^2}\\ &= \frac{A(1+t^2)+(Bt+C)t}{t(1+t^2)}\end{aligned}$$

(2M)

Hence $A+B=2$, $C=3$. $A=1$ and $B=1$. **(2A)**

So we have

$$\begin{aligned}\int_1^2 \frac{2t^2+3t+1}{t(t^2+1)}\,dt &= \int_1^2 \frac{1}{t}+\frac{t+3}{t^2+1}\,dt\\ &= [\ln|t|]_1^2+\int_1^2 \frac{t}{1+t^2}\,dt+\int_1^2 \frac{3}{1+t^2}\\ &= [\ln t+\frac{1}{2}\ln(1+t^2)+3\tan^{-1}t]_1^2\\ &= \ln 2+\frac{1}{2}\ln 5+3\tan^{-1}2-\frac{1}{2}\ln 2-3\tan^{-1}1\\ &= 2.12\end{aligned}$$

(3M, 1A (must be to 2 dp)

(8)

(B6)

$$\begin{aligned} 4x - y - z &= 2 \\ 2x + \alpha y + z &= 4 \\ x - 2y - 2z &= -3 \end{aligned}$$

Require $|A| = 0$ i.e.

$$\begin{aligned} \begin{vmatrix} 4 & -1 & -1 \\ 2 & \alpha & 1 \\ 1 & -2 & -2 \end{vmatrix} &= 4\begin{vmatrix} \alpha & 1 \\ -2 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & \alpha \\ 1 & -2 \end{vmatrix} \\ &= 4(-2\alpha + 2) + (-4 - 1) - (-4 - \alpha) = -7\alpha + 7 = 0 \end{aligned}$$

(2M)

Hence $\alpha = 1$. **(1a)**

Solve

$$\begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}.$$

Add rows 1 and 2 to get $6x = 6$ hence $x = 1$. From row 3 we have $-2y - 2z = -3 - x = -4 \rightarrow y = 2 - z$ Let $z = \lambda$ giving the final solution $(x, y, z) = (1, 2 - \lambda, \lambda)$. (or any other correct method that gives the right solution) **(2M, 1A)**

If $\alpha = -2$, there is an inverse so the solution is unique. The only solution to the homogeneous problem for this case is $(x, y, z) = (0, 0, 0)$ **(2A)**

(8)

(B7) $A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$

Need to solve $|A - \lambda I| = 0$.

$$\begin{aligned} \begin{vmatrix} -\lambda & 2 & 4 \\ 1 & 1-\lambda & -2 \\ -2 & 0 & 5-\lambda \end{vmatrix} &= -\lambda(1-\lambda)(5-\lambda) - 2(5-\lambda-4) + 4(2(1-\lambda)) \\ &= (1-\lambda)(\lambda^2 - 5\lambda - 2 + 8) = (1-\lambda)(\lambda-3)(\lambda-2) \end{aligned}$$

(2M)

Hence $\lambda = 1, 2, 3$ **(2A)**

For $\lambda = 1$ we need to solve

$$\begin{aligned} -x + 2y + 4z &= 0 \\ x - 2z &= 0 \\ -2x + 4z &= 0 \end{aligned}$$

(1M each case similar so only 1 method mark)

Hence $x = 2z$ and $y = -z$. So the eigenvector is of the form $\alpha \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

(1A)

For $\lambda = 2$ we need to solve

$$\begin{aligned} -2x + 2y + 4z &= 0 \\ x - y - 2z &= 0 \\ -2x + 3z &= 0 \end{aligned}$$

Hence $2x = 3z$ and $2y = -z$. So the eigenvector is of the form $\beta \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$.

(1A)

For $\lambda = 3$ we need to solve

$$\begin{aligned} -3x + 2y + 4z &= 0 \\ x - 2y - 2z &= 0 \\ -2x + 2z &= 0 \end{aligned}$$

Hence $x = z$ and $2y = -z$. So the eigenvector is of the form $\gamma \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$.

(1A)

Where $\alpha, \beta, \gamma \neq 0$ and are arbitrary.

(8)

(B8) First with Laplace Transforms

$$L(\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y)=L(\sin t)$$

Using the formula sheet this gives

$$s^2Y+3sY+2Y=\frac{1}{1+s^2}.$$

(1M, 1A)

So

$$\begin{aligned}
Y &= \frac{1}{1+s^2}\frac{1}{s^2+3s+2} \\
&= \frac{1}{(1+s^2)(s+2)(s+1)} \text{ use partial fractions} \\
&= \frac{As+b}{s^2+1}+\frac{C}{s+2}+\frac{D}{s+1} \\
&= \frac{(As+B)(s^2+3s+2)+C(s^2+1)(s+1)+D(s^2+1)(s+2)}{(1+s^2)(s+2)(s+1)} \\
&= \frac{As(s^2+3s+2)+B(s^2+3s+2)+C(s^3+s^2+s+1)+D(s^3+2s^2+s+2)}{(1+s^2)(s+2)(s+1)}
\end{aligned}$$

Equating power of s

$$\begin{aligned}
s^3: &\quad A+C+D=0 \\
s^2: &\quad 3A+B+C+2D=0 \\
s: &\quad 2A+3B+C+D=0 \\
1: &\quad 2B+C+2D=1
\end{aligned}$$

(3M)

Subtract first from third to get $A+3B=0\Rightarrow A=-3B$. Subtract 4th from 2nd to get $3A-B=-1\Rightarrow -9B-B=-1\Rightarrow B=\frac{1}{10}$ and $A=-\frac{3}{10}$. From first $C+D=\frac{3}{10}$ and from 4th $C+2D=1-\frac{2}{10}=\frac{4}{5}$.
Hence $D=\frac{1}{2},\ C=-\frac{1}{5}$.
Hence $Y=-\frac{3s}{10(s^2+1)}+\frac{1}{10(s^2+1)}-\frac{1}{5(s+2)}+\frac{1}{2(s+1)}$. **(2A)**
Use formula sheet to find the inverse Laplace transforms to get

$$y=-\frac{3}{10}\cos t+\frac{1}{10}\sin t-\frac{1}{5}e^{-2t}+\frac{1}{2}e^{-t}. \textbf{ (1A)}$$

OR

$$\begin{aligned} m^2 + 3m + 2 &= 0 \\ (m+2)(m+1) &= 0 \end{aligned}$$

(1M)
Hence $m = -2, -1$ and the complementary function is
$y_c = Ae^{-2t} + Be^{-t}$. **(1A)**
For the particular integral use

$$\begin{aligned} y_p &= a\cos t + b\sin t \\ \frac{dy}{dt} &= -a\sin t + b\cos t \\ \frac{d^2y}{dt^2} &= -a\cos t - b\sin t. \end{aligned}$$

(1M)
Hence

$$-a\cos t - b\sin t + 3(-a\sin t + b\cos t) + 2(a\cos t + b\sin t) = \sin t.$$

Equating coefficients of $\cos, \sin$

$$-a + 3b + 2z = 0 \Rightarrow a + 3b = 0 \Rightarrow a = -3b$$

$$-b - 3a + 2b = 1 \Rightarrow b - 3a = 1 \Rightarrow b + 9b = 1$$

hence $b = \frac{1}{10},\ a = -\frac{3}{10}$. **(2A)**
Hence

$$y = Ae^{-2t} + Be^{-t} - \frac{3}{10}\cos t + \frac{1}{10}\sin t$$

$$\frac{dy}{dt} = -2Ae^{-2t} - Be^{-t} + \frac{3}{10}\sin t + \frac{1}{10}\cos t$$

Using the initial conditions $y = 0$ and $\frac{dy}{dt} = 0$ at $t = 0$ we get $A + B = \frac{3}{10}$ and $2A + B = \frac{1}{10}$. Hence $A = -\frac{1}{5}$ and $B = \frac{1}{2}$. **(2M)**

$$y = -\frac{3}{10}\cos t + \frac{1}{10}\sin t - \frac{1}{5}e^{-2t} + \frac{1}{2}e^{-t}.$$

(1A)

(8)