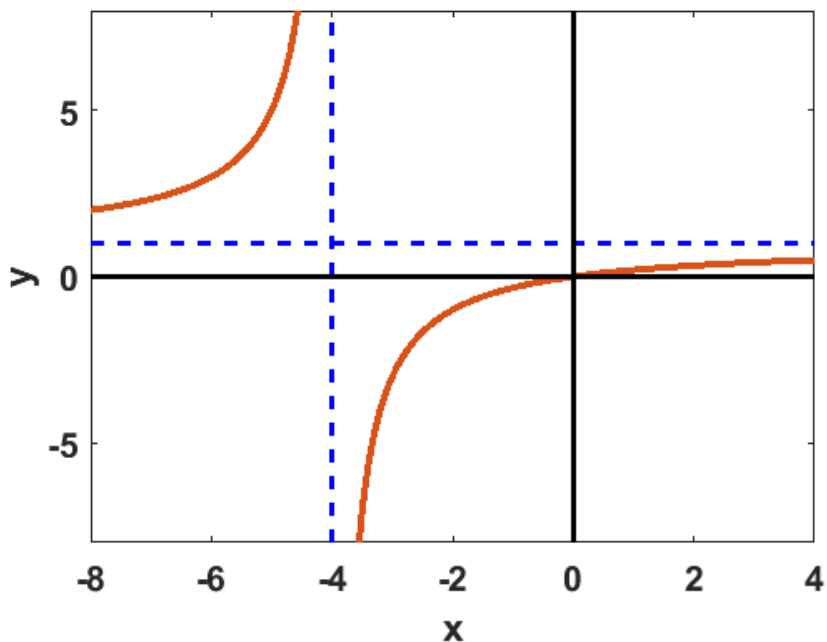


Solutions to MAS140\_151\_152\_156 2015-16

(A1)

$$\frac{x}{x+4}$$



Graph of  $y = \frac{x}{x+4}$ , dashed lines are asymptotes (1 mark), thick curve the function (1 mark). Must show these with x and y values clearly marked (1 mark).

(3)

(A2)  $y = e^{2x} - 1 \Rightarrow \ln(y + 1) = 2x \Rightarrow x = \frac{1}{2}\ln(y + 1)$ . (1M)

Hence  $f^{-1}(x) = \frac{1}{2}\ln(x + 1)$ . (1A)

Domain  $x > -1$ , range  $x \in \mathbb{R}$ . (1A)

(3)

(A3)  $f(x, y) = 4x^2\sqrt{y} + 5\cos(xy)$ .

$$\frac{\partial f}{\partial x} = 8x\sqrt{y} - 5y \sin(xy)$$

$$\frac{\partial f}{\partial y} = 4x^2 \frac{1}{2\sqrt{y}} - 5x \sin(xy) = \frac{2x^2}{\sqrt{y}} - 5x \sin(xy)$$
. (1M, 2A)

(3)

(A4) At  $x = 0$ ,  $x \tanh x = 0$  and  $\sin 2x = 0$ , hence need to apply l'Hôpital's Rule.

$$\lim_{x \rightarrow 0} \left( \frac{x \tanh x}{\sin 2x} \right) = \lim_{x \rightarrow 0} \left( \frac{\tanh x + x \operatorname{sech}^2 x}{2 \cos 2x} \right) = 0. \quad \text{(2M, 1A)} \quad (3)$$

(A5) Let  $z = a + ib$ . Since  $\operatorname{Re}(z) = \operatorname{Im}(z)$  we have  $z = a + ia = a(1 + i)$ .  
 $|z - 1 - i| = 1 \Rightarrow (a - 1)^2 + (a - 1)^2 = 2(a - 1)^2 = 1 \Rightarrow a - 1 = \pm \frac{1}{\sqrt{2}}$ .

Hence  $z = (1 \pm \frac{1}{\sqrt{2}})(1 + i)$ . (1M, 2A) (3)

(A6)  $\mathbf{a} = (7, -2, -5)$ ,  $\mathbf{b} = (5, 1, 3)$   $\mathbf{a} \cdot \mathbf{b} = 7 \times 5 + -2 \times 1 + -5 \times 3 = 35 - 2 - 15 = 18$ . (1A)

Vector perpendicular to  $\mathbf{a}$  and  $\mathbf{b}$  is  $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -2 & -5 \\ 5 & 1 & 3 \end{vmatrix}$

$$= \mathbf{i}(-2 \times 3 - (-5) \times 1) - \mathbf{j}(7 \times 3 - (-5) \times 5) + \mathbf{k}(7 \times 1 - (-2) \times 5)$$

$$= -\mathbf{i} - 46\mathbf{j} + 17\mathbf{k}. \quad \text{(1M, 1A)} \quad (3)$$

(A7)

$$\begin{aligned} \int_0^\pi (x + 1) \sin \frac{x}{2} dx &= \left[ (x + 1)(-2 \cos \frac{x}{2}) \right]_0^\pi - \int_0^\pi (-2 \cos \frac{x}{2}) dx \\ &= 2 + 2 \left[ 2 \sin \frac{x}{2} \right]_0^\pi = 2 + 4 = 6. \end{aligned}$$

(2M, 1A) (3)

$$(A8) \int x \cos^2 x^2 dx.$$

First make the substitution  $u = x^2$ ,  $\frac{du}{dx} = 2x$

$$\int x(\cos x^2)^2 dx = \int \frac{1}{2} \cos^2 u du.$$

Now use double angle formula  $\cos 2u = 2 \cos^2 u - 1$ .

$$\begin{aligned} \int \frac{1}{2} \cos^2 u du &= \int \frac{1}{4} (\cos 2u + 1) du \\ &= \frac{1}{4} \left( \frac{1}{2} \sin 2u + u \right) \\ &= \frac{1}{8} \sin 2x^2 + \frac{1}{4} x^2. \end{aligned}$$

**(2M (any), 1A)**

**(3)**

$$(A9) A^T = \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}. \quad \mathbf{(1A)}$$

$$(AB) = \begin{bmatrix} 1 \times 2 + 1 \times (-1) & 1 \times 1 + 1 \times 3 \\ 3 \times (-1) & 3 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 9 \end{bmatrix}.$$

$$(AB)^T = \begin{bmatrix} 1 & -3 \\ 4 & 9 \end{bmatrix} \quad \mathbf{(1A)}$$

$$B^T A^T = \begin{bmatrix} 2 \times 1 - 1 \times 1 & -1 \times 3 \\ 1 \times 1 + 3 \times 1 & 3 \times 3 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 9 \end{bmatrix} = (AB)^T. \quad \mathbf{(1A)}$$

**(3)**

(A10) Rewrite the equation as

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{e^x}{x^3}.$$

Integrating factor is  $= e^{\int \frac{3}{x} dx} = e^{3 \ln x} = e^{\ln x^3} = x^3$ . (1M)

Hence we have

$$\begin{aligned}\frac{d(yx^3)}{dx} &= e^x \\ yx^3 &= e^x + c \\ y &= \frac{e^x}{x^3} + \frac{c}{x^3}.\end{aligned}$$

(1M, 1A)

(3)

(A11) Write the equations in matrix form  $A\mathbf{x} = \mathbf{b}$  hence  $\mathbf{x} = A^{-1}\mathbf{b}$ .

$$\begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ -8 \end{pmatrix}.$$

$$|A| = -6 - 5 = -10 \text{ and } A^{-1} = \frac{-1}{10} \begin{pmatrix} -3 & -4 \\ -1 & 2 \end{pmatrix}. \text{ (2M)}$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{-1}{10} \begin{pmatrix} -3 & -4 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 14 \\ -8 \end{pmatrix} = \frac{-1}{10} \begin{pmatrix} -3 \times 14 - 4 \times (-8) \\ -14 - 2 \times 8 \end{pmatrix} = \frac{-1}{10} \begin{pmatrix} -10 \\ -30 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

(1A)

(3)

(A12) Looking for solutions of the form  $e^{mx}$ .

$$\text{Hence we have } m^2 + 2m + 14 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4-16}}{2} = -1 \pm \sqrt{3}i. \text{ (1M)}$$

So the general solution is of the form

$$y = e^{-x}(A \cos(\sqrt{3}x) + B \sin(\sqrt{3}x)).$$

When  $x = 0$ ,  $y = A$ , hence we must have  $A = 0$  and the general solution is

$$y = Be^{-x} \sin(\sqrt{3}x).$$

(1M, 1A)

(3)

(B1)

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 - 1$$

For stationary points

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y = 0$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y = 0$$

(2M)

Adding we have  $x^3 + y^3 = 0 \Rightarrow x = -y$  and substituting into the first equation (or otherwise)  $x^3 - 2x = 0 \Rightarrow x^2 = 0, 2$ .

Hence the stationary points are  $(0, 0), (\sqrt{2}, -\sqrt{2}), (-\sqrt{2}, \sqrt{2})$ . (3A)

We require the second derivatives evaluated at the stationary points:

$$f_{xx} = 12x^2 - 4 = 20 \text{ at both non-zero points}$$

$$f_{yy} = 12y^2 - 4 = 20 \text{ at both non-zero points}$$

$$f_{xy} = 4 \text{ everywhere}$$

(1M)

Hence  $\Delta = f_{xx}f_{yy} - (f_{xy})^2 = 20 \times 20 - 16 > 0$  and both points are minima. (1M, 1A)

(8)

(B2)

$$\begin{aligned}y &= \ln(1+x^2) = 0 @ x = 0 \\y' &= \frac{1}{1+x^2}(2x) = 0 @ x = 0 \\y'' &= \frac{2}{1+x^2} - \frac{2x \cdot 2x}{(1+x^2)^2} \\&= \frac{2(1+x^2-2x^2)}{(1+x^2)^2} \\&= \frac{2(1-x^2)}{(1+x^2)^2} = 2 @ x = 0 \\y''' &= \frac{-4x}{(1+x^2)^2} - \frac{4(1-x^2)2x}{(1+x^2)^3} \\&= \frac{4x(x^2-3)}{(1+x^2)^3} = 0 @ x = 0 \\y'''' &= 4 \frac{(x^2-3)+x(2x)}{(1+x^2)^3} - 12 \frac{x(x^2-3)2x}{(1+x^2)^4} \\&= 12 \frac{6x^2-x^4-1}{(1+x^2)^4} = -12 @ x = 0\end{aligned}$$

**(2M, 2a (0.5 each))**

Note that in the above it is not necessary to 'simplify' the expressions to determine the value of the derivatives at  $x = 0$ .

Hence the first two non-zero terms of the Maclaurin series are

$$y = \frac{2x^2}{2!} - \frac{12x^4}{4!} = x^2 - \frac{x^4}{2}.$$

**(1M, 1A)**

From the Formula Sheet  $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ .

Replacing  $x$  with  $x^2$  we have

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4}$$

and the first two terms are the same. **1M, 1A**

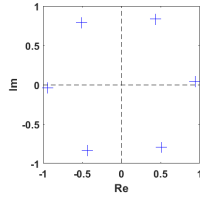
**(8)**

(B3)  $z_1 = 1 + i$ ,  $|z_1| = \sqrt{1+1} = \sqrt{2}$ ,  $\arg(z_1) = \frac{\pi}{4}$ . Hence  $z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$ . (1M, 1A)

$z_2 = \sqrt{3} + i$ ,  $|z_2| = \sqrt{3+1} = 2$ ,  $\arg(z_2) = \frac{\pi}{6}$ . Hence  $z_2 = 2e^{i\frac{\pi}{6}}$ . (1M, 1A)

Hence  $z^6 = \frac{z_1}{z_2} = \frac{\sqrt{2}}{2}e^{i(\frac{\pi}{4}-\frac{\pi}{6})} = \frac{1}{\sqrt{2}}e^{i(\frac{\pi}{12}+2\pi n)}$ .

Hence  $z = \frac{1}{2^{1/2}}e^{i(\frac{\pi}{72}+\frac{n\pi}{3})}$  for  $n = 1, \dots, 6$ . (2M, 1A) The six solutions



are marked with + in the figure. The plot does not have to be exact but should make it clear that they lie on a circle (same modulus) separated by  $\frac{\pi}{3}$  in argument with none on the Re or Im axes. (1A)

(8)

(B4) (a)

$$\mathbf{r}(t) = (2t^2, t^2 - 4t, 3t - 5)$$

$$\frac{d\mathbf{r}}{dt} = (4t, 2t - 4, 3)$$

$$\frac{d^2\mathbf{r}}{dt^2} = (4, 2, 0)$$

(2A)

(b) unit vector, at  $t = 1$ ,  $\hat{r}(t) = \frac{(2, -3, -2)}{\sqrt{4+9+4}} = \frac{(2, -3, -2)}{\sqrt{17}}$ . (1M, 1A)

Hence components of velocity and acceleration in this direction are:

$$\frac{d\mathbf{r}}{dt} \cdot \hat{r}(t) = (4, -2, 3) \cdot \frac{(2, -3, -2)}{\sqrt{17}} = \frac{8}{\sqrt{17}}$$

$$\frac{d^2\mathbf{r}}{dt^2} \cdot \hat{r}(t) = (4, 2, 0) \cdot \frac{(2, -3, -2)}{\sqrt{17}} = \frac{2}{\sqrt{17}} = \frac{1}{4} \frac{8}{\sqrt{17}}$$

(2M, 2A)

(8)

(B5)

$$\int_1^2 \frac{2t^2 + 3t + 1}{t^3 + t} dt = \int_1^2 \frac{2t^2 + 3t + 1}{t(t^2 + 1)} dt$$

Need to use partial fractions.

$$\begin{aligned} \frac{2t^2 + 3t + 1}{t(t^2 + 1)} &= \frac{A}{t} + \frac{Bt + C}{1 + t^2} \\ &= \frac{A(1 + t^2) + (Bt + C)t}{t(1 + t^2)} \end{aligned}$$

(2M)

Hence  $A + B = 2$ ,  $C = 3$ .  $A = 1$  and  $B = 1$ . (2A)

So we have

$$\begin{aligned} \int_1^2 \frac{2t^2 + 3t + 1}{t(t^2 + 1)} dt &= \int_1^2 \frac{1}{t} + \frac{t + 3}{t^2 + 1} dt \\ &= [\ln |t|]_1^2 + \int_1^2 \frac{t}{1 + t^2} dt + \int_1^2 \frac{3}{1 + t^2} \\ &= [\ln t + \frac{1}{2} \ln(1 + t^2) + 3 \tan^{-1} t]_1^2 \\ &= \ln 2 + \frac{1}{2} \ln 5 + 3 \tan^{-1} 2 - \frac{1}{2} \ln 2 - 3 \tan^{-1} 1 \\ &= 2.12 \end{aligned}$$

(3M, 1A (must be to 2 dp)

(8)



(B6)

$$\begin{aligned}4x - y - z &= 2 \\2x + \alpha y + z &= 4 \\x - 2y - 2z &= -3\end{aligned}$$

Require  $|A| = 0$  i.e.

$$\begin{aligned}\begin{vmatrix} 4 & -1 & -1 \\ 2 & \alpha & 1 \\ 1 & -2 & -2 \end{vmatrix} &= 4 \begin{vmatrix} \alpha & 1 \\ -2 & -2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} - 1 \begin{vmatrix} 2 & \alpha \\ 1 & -2 \end{vmatrix} \\ &= 4(-2\alpha + 2) + (-4 - 1) - (-4 - \alpha) = -7\alpha + 7 = 0\end{aligned}$$

(2M)

Hence  $\alpha = 1$ . (1a)

Solve

$$\begin{pmatrix} 4 & -1 & -1 \\ 2 & 1 & 1 \\ 1 & -2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}.$$

Add rows 1 and 2 to get  $6x = 6$  hence  $x = 1$ . From row 3 we have  $-2y - 2z = -3 - x = -4 \rightarrow y = 2 - z$  Let  $z = \lambda$  giving the final solution  $(x, y, z) = (1, 2 - \lambda, \lambda)$ . (or any other correct method that gives the right solution) (2M, 1A)

If  $\alpha = -2$ , there is an inverse so the solution is unique. The only solution to the homogeneous problem for this case is  $(x, y, z) = (0, 0, 0)$  (2A)

(8)

$$(B7) \quad A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$$

Need to solve  $|A - \lambda I| = 0$ .

$$\begin{aligned} \begin{vmatrix} -\lambda & 2 & 4 \\ 1 & 1-\lambda & -2 \\ -2 & 0 & 5-\lambda \end{vmatrix} &= -\lambda(1-\lambda)(5-\lambda) - 2(5-\lambda-4) + 4(2(1-\lambda)) \\ &= (1-\lambda)(\lambda^2 - 5\lambda - 2 + 8) = (1-\lambda)(\lambda-3)(\lambda-2) \end{aligned}$$

**(2M)**

Hence  $\lambda = 1, 2, 3$  **(2A)**

For  $\lambda = 1$  we need to solve

$$\begin{aligned} -x + 2y + 4z &= 0 \\ x - 2z &= 0 \\ -2x + 4z &= 0 \end{aligned}$$

**(1M each case similar so only 1 method mark)**

Hence  $x = 2z$  and  $y = -z$ . So the eigenvector is of the form  $\alpha \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ .

**(1A)**

For  $\lambda = 2$  we need to solve

$$\begin{aligned} -2x + 2y + 4z &= 0 \\ x - y - 2z &= 0 \\ -2x + 3z &= 0 \end{aligned}$$

Hence  $2x = 3z$  and  $2y = -z$ . So the eigenvector is of the form  $\beta \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$ .

**(1A)**

For  $\lambda = 3$  we need to solve

$$\begin{aligned} -3x + 2y + 4z &= 0 \\ x - 2y - 2z &= 0 \\ -2x + 2z &= 0 \end{aligned}$$

Hence  $x = z$  and  $2y = -z$ . So the eigenvector is of the form  $\gamma \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$ .

**(1A)**

Where  $\alpha, \beta, \gamma \neq 0$  and are arbitrary.

**(8)**

(B8) First with Laplace Transforms

$$L\left(\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y\right) = L(\sin t)$$

Using the formula sheet this gives

$$s^2Y + 3sY + 2Y = \frac{1}{1 + s^2}.$$

(1M, 1A)

So

$$\begin{aligned} Y &= \frac{1}{1 + s^2} \frac{1}{s^2 + 3s + 2} \\ &= \frac{1}{(1 + s^2)(s + 2)(s + 1)} \text{ use partial fractions} \\ &= \frac{As + b}{s^2 + 1} + \frac{C}{s + 2} + \frac{D}{s + 1} \\ &= \frac{(As + B)(s^2 + 3s + 2) + C(s^2 + 1)(s + 1) + D(s^2 + 1)(s + 2)}{(1 + s^2)(s + 2)(s + 1)} \\ &= \frac{As(s^2 + 3s + 2) + B(s^2 + 3s + 2) + C(s^3 + s^2 + s + 1) + D(s^3 + 2s^2 + s + 2)}{(1 + s^2)(s + 2)(s + 1)} \end{aligned}$$

Equating power of  $s$

$$\begin{aligned} s^3: & \quad A + C + D = 0 \\ s^2: & \quad 3A + B + C + 2D = 0 \\ s: & \quad 2A + 3B + C + D = 0 \\ 1: & \quad 2B + C + 2D = 1 \end{aligned}$$

(3M)

Subtract first from third to get  $A + 3B = 0 \Rightarrow A = -3B$ . Subtract 4th from 2nd to get  $3A - B = -1 \Rightarrow -9B - B = -1 \Rightarrow B = \frac{1}{10}$  and  $A = -\frac{3}{10}$ . From first  $C + D = \frac{3}{10}$  and from 4th  $C + 2D = 1 - \frac{2}{10} = \frac{4}{5}$ . Hence  $D = \frac{1}{2}$ ,  $C = -\frac{1}{5}$ .

$$\text{Hence } Y = -\frac{3s}{10(s^2+1)} + \frac{1}{10(s^2+1)} - \frac{1}{5(s+2)} + \frac{1}{2(s+1)}. \quad \text{(2A)}$$

Use formula sheet to find the inverse Laplace transforms to get

$$y = -\frac{3}{10} \cos t + \frac{1}{10} \sin t - \frac{1}{5} e^{-2t} + \frac{1}{2} e^{-t}. \quad \text{(1A)}$$

OR

$$\begin{aligned}m^2 + 3m + 2 &= 0 \\(m + 2)(m + 1) &= 0\end{aligned}$$

(1M)

Hence  $m = -2, -1$  and the complementary function is

$$y_c = Ae^{-2t} + Be^{-t}. \quad (1A)$$

For the particular integral use

$$\begin{aligned}y_p &= a \cos t + b \sin t \\ \frac{dy}{dt} &= -a \sin t + b \cos t \\ \frac{d^2y}{dt^2} &= -a \cos t - b \sin t.\end{aligned}$$

(1M)

Hence

$$-a \cos t - b \sin t + 3(-a \sin t + b \cos t) + 2(a \cos t + b \sin t) = \sin t.$$

Equating coefficients of  $\cos, \sin$

$$\begin{aligned}-a + 3b + 2a &= 0 \Rightarrow a + 3b = 0 \Rightarrow a = -3b \\ -b - 3a + 2b &= 1 \Rightarrow b - 3a = 1 \Rightarrow b + 9b = 1\end{aligned}$$

$$\text{hence } b = \frac{1}{10}, a = -\frac{3}{10}. \quad (2A)$$

Hence

$$y = Ae^{-2t} + Be^{-t} - \frac{3}{10} \cos t + \frac{1}{10} \sin t$$

$$\frac{dy}{dt} = -2Ae^{-2t} - Be^{-t} + \frac{3}{10} \sin t + \frac{1}{10} \cos t$$

Using the initial conditions  $y = 0$  and  $\frac{dy}{dt} = 0$  at  $t = 0$  we get  $A + B = \frac{3}{10}$  and  $2A + B = \frac{1}{10}$ . Hence  $A = -\frac{1}{5}$  and  $B = \frac{1}{2}$ . (2M)

$$y = -\frac{3}{10} \cos t + \frac{1}{10} \sin t - \frac{1}{5} e^{-2t} + \frac{1}{2} e^{-t}.$$

(1A)

(8)