University of Sheffield

School of Mathematics and Statistics

MAS140/151/152/156 Engineering Mathematics

Spring Semester

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Outline Syllabus: Spring Semester

- **Integration:** Indefinite integrals of simple functions. Simple substitutions. Standard forms involving inverse trigonometric and inverse hyperbolic functions. Examples using completing the square and partial fractions. Integration by parts. Definite integrals: properties, evaluation, application to area.

- **Matrices and linear equations:** Definition of an $m \times n$ matrix. Special matrices (identity, zero, square etc.). Matrix algebra. Transpose. Symmetric and skew-symmetric matrices, and the decomposition of square matrices. Determinants. Inverse of a non-singular matrix. Use of matrices to solve systems of linear equations (homogeneous and nonhomogeneous). Gaussian elimination. Eigenvalues and eigenvectors.

- **Ordinary differential equations:** First order differential equations: variables separable, linear with integrating factor, general solution, solution satisfying given initial conditions. Second order linear differential equations with constant coefficients: auxiliary equation, complementary function. Particular integral for polynomials, exponentials, trigonometric functions and products of polynomials and exponential/trigonometric functions on right-hand side. Laplace transforms and application to the solution of ODEs.

1 Integration

1.1 Indefinite Integrals

A function $F(x)$ such that $\dfrac{dF}{dx} = f(x)$ is called an **indefinite integral** of $f(x)$.

We use the notation $\displaystyle\int f(x)\,dx = F(x)$.

Note that the dx is **not** optional in this notation. So (indefinite) integration is the inverse of differentiation.
If $F(x)$ is an indefinite integral of $f(x)$ then so is $F(x) + C$, where C is any constant, since $\dfrac{dC}{dx} = 0$. We should therefore always write

$$\int f(x)\,dx = F(x) + C. \tag{1.1}$$

We call C a **constant of integration**. In Equation (1.1), the function $f(x)$ is referred to as the **integrand**.

Note that if $\dfrac{dF}{dx} = f(x)$ and $\dfrac{dG}{dx} = g(x)$, then if we define

$$H_{\pm}(x) = F(x) \pm G(x) \text{ and } h_{\pm}(x) = f(x) \pm g(x),$$

then

$$\frac{dH_{\pm}}{dx} = h_{\pm}(x) \quad \Rightarrow \quad \int h_{\pm}(x)\,dx = H_{\pm}(x).$$

Therefore we have the useful result that

$$\int [f(x) \ \pm g(x)] \ dx = \int f(x)\,dx \ \pm \int g(x)\,dx. \tag{1.2}$$

Also, since $\dfrac{d(\alpha F)}{dx} = \alpha f(x)$ for any constant α, we have the result

$$\int \alpha f(x) dx = \alpha \int f(x)\,dx. \tag{1.3}$$

1.2 Integration of standard functions — examples

Since we know the derivatives of a range of standard functions, we can use these to infer a number of standard integrals.

1. If $F(x) = \dfrac{x^{n+1}}{n+1}$, where $n \neq -1$, then $\dfrac{dF}{dx} = x^n$ and so

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \qquad (n \neq -1).$$

2. For $x > 0$ we have $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ and so

$$\int \frac{1}{x} \, dx = \ln x + C.$$

The domain of $\ln x$ is $(0, \infty)$, so this only makes sense for $x > 0$. To allow for negative values of x, we write

$$\int \frac{dx}{x} = \ln |x| + C,$$

since the domain of $\ln |x|$ is $(-\infty, 0) \cup (0, \infty)$, and the derivative of $\ln |x|$ is $\dfrac{1}{x}$ on its entire domain. This distinction becomes important when considering definite integrals (see Sec. 1.4).

3. If $f(x) = \sin x$, then $F(x) = -\cos x$, because

$$\begin{aligned} &\frac{d}{dx}(-\cos x) = -(-\sin x) = \sin x \\ &\Rightarrow \int \sin x \, dx = -\cos x + C. \end{aligned}$$

4. If $f(x) = \sin 3x$, then $F(x) = -\frac{1}{3} \cos 3x$, because

$$\begin{aligned} &\frac{d}{dx}(\cos 3x) = (-\sin 3x)3 \\ &\Rightarrow \int \sin 3x \, dx = -\frac{1}{3} \cos 3x + C. \end{aligned}$$

5. If $f(x) = \sec^2\left(\dfrac{x}{2}\right)$, then $F(x) = 2\tan\left(\dfrac{x}{2}\right)$, because

$$\frac{d}{dx}\left[2\tan\left(\frac{x}{2}\right)\right] = 2\sec^2\left(\frac{x}{2}\right)\frac{1}{2} = \sec^2\left(\frac{x}{2}\right)$$

$$\Rightarrow \int \sec^2\left(\frac{x}{2}\right)\,dx = 2\tan\left(\frac{x}{2}\right) + C.$$

6. $$\int \cos^2 x\,dx = \frac{1}{2}\int (\cos 2x + 1)\,dx$$

(since $\cos 2x = 2\cos^2 x - 1$)

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + C.$$

Exercise: Find indefinite integrals of each of the following

$\cos(4x)$, $\cosh^2 x$, $\operatorname{sech}^2\left(\frac{x}{4}\right)$, $\sinh^2\left(\frac{x}{2}\right)$, e^{2x}.

1.3 Methods of Integration

Given our knowledge of some standard integrals, we can tackle a range of integrals whose integrands have particular forms using a few standard techniques. The key to becoming good at integration is to be able to spot which techniques will be helpful in any particular situation. This facility can only really be gained by doing lots of examples . . .

1.3.1 Integration by Substitution

To evaluate $F(x)$ it is often possible to make a change of variable to get a simpler problem. If we put $x = x(t)$ then, by the function-of-a-function rule (chain rule),

$$\frac{dF}{dt} = \frac{dF}{dx}\frac{dx}{dt} = f(x)\frac{dx}{dt} = f\,[\;x(t)\;]\,\frac{dx}{dt}.$$

Integrating the left and right parts of the above with respect to t we can write

$$F(t) = \int f\,[\;x(t)\;]\,\frac{dx}{dt}\,dt. \tag{1.4}$$

Thus, if we have an integrand that takes the form in Equation (1.4), then we can use a change of variables (from x to t) to evaluate the integral as a function of the new variable t. It is usually necessary (or at least desirable) to write the solution in terms of the original variable x. This requires $x = x(t)$ to be rearranged to give $t = t(x)$; the result can then be substituted into $F(t)$.

Example: Use the substitution $x = \tan\theta$ to show that $\displaystyle\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$

Solution: If $x = \tan\theta$ then $\dfrac{dx}{d\theta} = \sec^2\theta$ and $1 + x^2 = 1 + \tan^2\theta$, and so

$$\begin{aligned}
\int \frac{dx}{1+x^2} &= \int \frac{1}{1+\tan^2\theta}\sec^2\theta\,d\theta \\
&= \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta \\
&= \int d\theta \\
&= \theta + C.
\end{aligned}$$

To return to the original variable, x, it is necessary to eliminate θ. Since $x = \tan\theta$, $\theta = \tan^{-1} x$, and so we obtain the desired result:

$$\int \frac{dx}{1+x^2} = \tan^{-1} x + C.$$

Note that we could actually have obtained the last result directly from our knowledge that

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2},$$

because integrating both sides gives

$$\begin{aligned}
&\tan^{-1} x = \int \frac{dx}{1+x^2} + \text{constant} \\
\Rightarrow \qquad &\int \frac{dx}{1+x^2} = \tan^{-1} x + C.
\end{aligned}$$

Integrals of the type $\displaystyle\int f[g(x)]\, g'(x)\, dx$ are evaluated by using the substitution $u = g(x)$. Using this substitution, we have

$$I = \int f[g(x)]\, g'(x)\, dx = \int f(u)\, \frac{du}{dx}\, dx.$$

Now, we can use the chain rule to write $\dfrac{du}{dx}\, dx = du$ (an infinitesimal change dx results in a corresponding infinitesimal change du).
Therefore

$$I = \int f(u)\, du = F(u) + C,$$

where $F(u)$ is a function such that $\dfrac{dF}{du} = f(u)$ and C is a constant of integration.

Example: Evaluate $\displaystyle\int 2xe^{x^2}\, dx$

Solution: Let $f(s) = e^s$ and $g(x) = x^2$, then $g'(x) = 2x$ and therefore

$$f[g(x)]g'(x) = e^{g(x)}g'(x) = e^{x^2}2x.$$

Hence using the substitution $u = g(x) = x^2$, we have $du = 2xdx$ and so

$$\begin{aligned}
\int 2xe^{x^2}\, dx &= \int e^u\, du \\
&= e^u + C \\
&= e^{x^2} + C.
\end{aligned}$$

Exercise: Evaluate $\displaystyle\int \cosh^2 x \sinh x\, dx$.

Solution: $\cosh^2 x \sinh x$ is of the form $f[g(x)]g'(x)$ with

$$f(x) = x^2 \qquad \text{and} \qquad g(x) = \cosh x.$$

Now let $u = g(x) = \cosh x$, then $\dfrac{du}{dx} = \sinh x$ or $du = \sinh x dx$ and so

$$\begin{aligned}\int \cosh^2 x \sinh x \, dx &= \int u^2 \, du \qquad (\text{ where } u = \cosh x) \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\cosh^3 x + C.\end{aligned}$$

Special Case: Note that if $f[g(x)] = 1/g(x)$ then

$$\int \frac{g'(x)}{g(x)} \, dx = \ln g(x) + C,$$

as may be seen by using the substitution $u = g(x)$, so that $du = g'(x)dx$. Thus

$$\begin{aligned}\int \frac{g'(x)}{g(x)} \, dx &= \int \frac{du}{u} \\ &= \ln u + C \\ &= \ln g(x) + C.\end{aligned}$$

Example: Find $\displaystyle\int \frac{x}{2+3x^2} \, dx$.

Solution: The integrand is "nearly" in the required form, $(g'(x)/g(x))$, but the numerator differs from being the derivative of the denominator by a factor of 6.
Let $u = 2 + 3x^2$, then $du = 6xdx$ and so

$$\begin{aligned}\int \frac{x}{2+3x^2} \, dx &= \int \frac{1}{u}\left(\frac{du}{6}\right) \\ &= \frac{1}{6}\ln u + C \\ &= \frac{1}{6}\ln\left(2+3x^2\right) + C.\end{aligned}$$

Example: Find $\displaystyle\int \frac{1}{1-x^2} \, dx$.

Solution: To solve this problem we need to rewrite the integrand using **partial fractions**. We need to learn/revise this before proceeding.

Since $1 - x^2 = (1-x)(1+x)$ we can write

$$\frac{1}{1-x^2} \equiv \frac{A}{1-x} + \frac{B}{1+x}$$

for some constants A and B (which we aim to determine). Multiplying through by $1 - x^2$ gives

$$1 \equiv A(1+x) + B(1-x)$$

Putting $x = 1 \Rightarrow 1 = 2A \Rightarrow A = 1/2$
Putting $x = -1 \Rightarrow 1 = 2B \Rightarrow B = 1/2$
Thus

$$\int \frac{dx}{1-x^2} = \int \left(\frac{1}{2(1-x)} + \frac{1}{2(1+x)} \right) dx = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x},$$

where we have used the result derived in Equation (1.2). We could now use a substitution to evaluate these integrals (e.g. $u = 1 - x$, $du = -1dx$), but we can go straight to the solution:

$$\begin{aligned} \int \frac{dx}{1-x^2} &= -\frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) + C \\ &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + C. \end{aligned}$$

As in the above example, it is possible here to spot the solution without making an explicit substitution. Being able to spot such "short-cuts" (and implement them correctly!) comes with practice. Make use of the Examples Sheets ...

Exercise: Find $I(x) = \displaystyle\int \frac{x+3}{x^2 - x - 6}\, dx.$

Solution: Factorising $x^2 - x - 6 = (x+2)(x-3)$ we have that

$$\frac{x+3}{x^2-x-6} \equiv \frac{A}{x+2} + \frac{B}{x-3} \quad \Rightarrow \quad x + 3 \equiv A(x-3) + B(x+2)$$

Putting $x = 3 \Rightarrow 6 = 5B \Rightarrow B = 6/5$

Putting $x = -2 \Rightarrow 1 = -5A \Rightarrow A = -1/5$

Thus $I(x) = \displaystyle\int \frac{x+3}{x^2-x-6}\, dx = -\frac{1}{5}\int \frac{dx}{x+2} + \frac{6}{5}\int \frac{dx}{x-3}$

As above, we could make explicit substitutions, or we can skip straight to the solution:

$$
\begin{aligned}
I(x) &= -\frac{1}{5}\ln(x+2) + \frac{6}{5}\ln(x-3) + C \\
&= \frac{1}{5}[\ln(x+2)^{-1} + \ln(x-3)^6] + C \\
&= \frac{1}{5}\ln\left[\frac{(x-3)^6}{x+2}\right] + C,
\end{aligned}
$$

where we have made use of the standard properties of the $\ln$ function.

Special Case: To evaluate integrals with integrands that transform, after an appropriate substitution, transform into one of the following:

$$
\int \sqrt{a^2+u^2}\,du, \qquad \int \sqrt{a^2-u^2}\,du, \qquad \int \sqrt{u^2-a^2}\,du,
$$

we use trigonometric or hyperbolic function substitutions, and appropriate trigonometric or hyperbolic relationships to simplify the resulting integral.

Example: Evaluate $I(x) = \displaystyle\int \sqrt{4-x^2}\,dx$

Solution: Use the substitution $x = 2\sin\theta$, so that $dx = 2\cos\theta d\theta$ and

$$
4 - x^2 = 4 - 4\sin^2\theta = 4(1-\sin^2\theta) = 4\cos^2\theta.
$$

Hence $\sqrt{4-x^2} = 2\cos\theta$ and

$$
\begin{aligned}
I(x) = \int \sqrt{4-x^2}\,dx &= \int 2\cos\theta(2\cos\theta\,d\theta) \\
&= 4\int \cos^2\theta\,d\theta \\
&= 4\int \frac{1}{2}(1+\cos 2\theta)\,d\theta \qquad [\text{using } \cos 2\theta = 2\cos^2\theta - 1] \\
&= 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= 2\theta + \sin 2\theta + C \\
&= 2\theta + 2\sin\theta\cos\theta + C
\end{aligned}
$$

As in earlier examples, we now seek to eliminate θ to obtain the required integral in terms of x. Using $x = 2\sin\theta$, we have that

$$
\begin{aligned}
I(x) &= 2\sin^{-1}\left(\frac{x}{2}\right) + 2\frac{x}{2}\sqrt{1-\left(\frac{x}{2}\right)^2} + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x}{2}\sqrt{4-x^2} + C.
\end{aligned}
$$

Exercise: Find $I(x) = \displaystyle\int \sqrt{9-x^2}\, dx$.

Solution: Use the substitution $x = 3\sin\theta$, so that $dx = 3\cos\theta d\theta$ and

$$9 - x^2 = 9 - 9\text{sin}^2\theta = 9(1-\text{sin}^2\theta) = 9\text{cos}^2\theta.$$

Hence $\sqrt{9-x^2} = 3\cos\theta$ and

$$\begin{aligned}
I(x) = \int \sqrt{9-x^2}\, dx &= \int 3\cos\theta(3\cos\theta\, d\theta) \\
&= 9\int \cos^2\theta\, d\theta \\
&= 9\int \frac{1}{2}(1+\cos 2\theta)\, d\theta \qquad \text{[using } \cos 2\theta = 2\cos^2\theta - 1\text{]} \\
&= \frac{9}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + C \\
&= \frac{9}{2}\left(\theta + \sin\theta\cos\theta\right) + C \\
&= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{3}\sqrt{1-\left(\frac{x}{3}\right)^2}\right) + C \\
&= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x}{2}\sqrt{9-x^2} + C.
\end{aligned}$$

In the examples above, $x^2 < a^2$ (where $a = 2$ in the example and $a = 3$ in the exercise). We simplified $\sqrt{a^2-x^2}$ by using the identity $\text{cos}^2\theta + \text{sin}^2\theta = 1$.
If instead the integrand is of the form $\sqrt{x^2-a^2}$ (where we require $x^2 > a^2$), we use a hyperbolic substitution and the hyperbolic identity $\text{cosh}^2\theta - \text{sinh}^2\theta = 1$. The working is analogous to the examples above. The same technique is also used for integrands of the form $\sqrt{x^2+a^2}$.

Exercise: Find $I(x) = \displaystyle\int \sqrt{x^2-4}\, dx$.

Solution: Let $x = 2\cosh\theta$ so that

$$dx = 2\sinh\theta d\theta \quad \text{and} \quad x^2 - 4 = 4\text{cosh}^2\theta - 4 = 4(\text{cosh}^2\theta - 1) = 4\text{sinh}^2\theta.$$

Hence

$$\begin{aligned}
I(x) = \int \sqrt{x^2-4}\, dx &= 4\int \sinh^2\theta\, d\theta \\
&= 2\int (\cosh 2\theta - 1)\, d\theta \qquad \text{[since } \cosh 2\theta = 1 + 2\sinh^2\theta\text{]} \\
&= 2\left(\frac{1}{2}\sinh 2\theta - \theta\right) + C \\
&= 2\sinh\theta\cosh\theta - 2\theta + C \qquad \text{[since } \sinh 2\theta = 2\sinh\theta\cosh\theta\text{]}
\end{aligned}$$

Now we need to write this in terms of the original variable, x. We need to use:

$$\begin{aligned}
\cosh\theta &= \frac{x}{2} \quad \Rightarrow \quad \sinh\theta = \sqrt{\cosh^2\theta - 1} = \sqrt{\frac{x^2}{4} - 1} = \frac{1}{2}\sqrt{x^2-4}. \\
\text{So } I(x) &= 2\sinh\theta\cosh\theta - 2\theta + C \\
&= \sqrt{(x^2-4)}\frac{x}{2} - 2\cosh^{-1}\left(\frac{x}{2}\right) + C \\
&= \frac{1}{2}x\sqrt{x^2-4} - 2\cosh^{-1}\left(\frac{x}{2}\right) + C.
\end{aligned}$$

Sometimes the integrand isn't quite in the form required to apply the substitutions detailed above directly, but can be cast in this form with a little preliminary work.

Example: Find $I(x) = \displaystyle\int \sqrt{3-2x-x^2}\,dx$.

Solution: To solve this problem we need to rewrite the integrand by **completing the square**. We need to learn/revise this before proceeding.

$$\begin{aligned}
3-2x-x^2 &= -(x^2+2x)+3 \\
&= -[(x+1)^2-1]+3 \\
&= 4-(x+1)^2.
\end{aligned}$$

The integrand is now very similar to that in a previous example, with x replaced by $x+1$. We therefore use the substitution $x+1=u$ so that $dx = du$. Then we have

$$I(x) = \int \sqrt{4-u^2}\,du.$$

We know from our previous example that

$$\int \sqrt{4-u^2}\,du = 2\sin^{-1}\left(\frac{u}{2}\right) + \frac{u}{2}\sqrt{4-u^2} + C,$$

where we have simply replaced the variable x in the earlier example by the variable u. Substituting $u = x+1$, we obtain the required answer:

$$\begin{aligned}
I(x) &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{4-(x+1)^2} + C \\
&= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C.
\end{aligned}$$

Other trigonometric and hyperbolic substitutions:

Integrals which, after an appropriate substitution transform into one of the following:

$$\int \frac{du}{a^2+u^2}, \quad \int \frac{du}{a^2-u^2}, \quad \int \frac{du}{\sqrt{a^2+u^2}}, \quad \int \frac{du}{\sqrt{a^2-u^2}}, \quad \int \frac{du}{\sqrt{u^2-a^2}}$$

can be found by using an appropriate trigonometric or hyperbolic substitution, and a corresponding identity (using an analogous approach to that used in the examples above).

Example: Show that $I(x) = \displaystyle\int \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$

Solution: We use the substitution $x = 2\tan\theta$, so that $dx = 2\sec^2\theta\, d\theta$ and

$$4 + x^2 = 4 + 4\tan^2\theta = 4\left(1+\tan^2\theta\right) = 4\sec^2\theta.$$

Substituting in, we obtain

$$I(x) = \int \frac{1}{2}\, d\theta = \frac{\theta}{2} + C = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Exercise: Show that (to within an arbitrary constant),

Integral	Substitution
$\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$	$u = a\tan\theta$
$\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\left(\frac{u}{a}\right)$	$u = a\sin\theta$
$\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right)$	$u = a\cosh\theta$
$\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right)$	$u = a\sinh\theta$

Example: Find $I(x) = \displaystyle\int \frac{dx}{\sqrt{4+9x^2}}.$

Solution: We need to make a substitution to get the integrand into one of the forms above. Let $u = 3x$ so that $du = 3dx$, then

$$\begin{aligned} I(x) = \int \frac{dx}{\sqrt{4+9x^2}} &= \frac{1}{3}\int \frac{du}{\sqrt{2^2+u^2}} \\ &= \frac{1}{3}\sinh^{-1}\left(\frac{u}{2}\right) + C \\ &= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{2}\right) + C \end{aligned}$$

Exercise: Find $I(x) = \displaystyle\int \frac{dx}{x^2+2x+3}.$

Solution: We begin by completing the square:

$$x^2 + 2x + 3 = (x+1)^2 + 2.$$

So we can use the substitution $u = x + 1 \Rightarrow dx = du$. Then

$$I(x) = \int \frac{dx}{x^2 + 2x + 3} = \int \frac{du}{u^2 + 2} = \int \frac{du}{u^2 + \left[\sqrt{2}\right]^2}$$
$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C.$$

Example: Find $I(x) = \displaystyle\int \frac{\cos x dx}{\sqrt{1 + \sin^2 x}}$.

Solution: We use the substitution $u = \sin x \Rightarrow du = \cos x dx$. Then

$$\begin{aligned} I(x) = \int \frac{\cos x\, dx}{\sqrt{1 + \sin^2 x}} &= \int \frac{du}{\sqrt{1 + u^2}} \\ &= \sinh^{-1} u + C \\ &= \sinh^{-1}(\sin x) + C. \end{aligned}$$

Rational functions of $\sin x$ and $\cos x$

If the integrand is a rational function of $\sin x$ and $\cos x$ (for example, $\dfrac{3\cos x}{5 \sin x + 2\cos x + 1}$), the integral can usually be evaluated by the substitution

$$t = \tan(x/2).$$

We can then make use of the trigonometric relations

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2},$$
$$\tan x = \frac{2t}{1-t^2}, \qquad \frac{dx}{dt} = \frac{2}{1+t^2}.$$

If the integrand contains $\sin x$ and $\cos x$ in the form $\cos^2 x, \sin^2 x$ or $\sin x \cos x$ only, then it can be written in terms of $\sin 2x$ and $\cos 2x$. In this case $t = \tan x$ will be a better substitution.

Rational functions of $\sinh x$ and $\cosh x$

If the integrand is a rational function of $\sinh x$ and $\cosh x$, the integral can usually be evaluated by the substitution

$$t = \tanh(x/2).$$

We can then make use of the hyperbolic relations

$$\sinh x = \frac{2t}{1-t^2}, \qquad \cosh x = \frac{1+t^2}{1-t^2},$$
$$\tanh x = \frac{2t}{1+t^2}, \qquad \frac{dx}{dt} = \frac{2}{1-t^2}.$$

1.3.2 Integration by Parts

If $u = u(x), v = v(x)$ then, by the product rule for differentiation, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Integrating both sides with respect to x gives

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx,$$

the constant of integration being absorbed into the integrals. This equation may be re-written as

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx, \tag{1.5}$$

which is the formula for **integration by parts**. If we wish to evaluate an integral where the integrand is in the form of the product of two functions, then we aim to use Equation (1.5) to *simplify* the problem. We therefore need to decide how best to identify parts of the integrand with u and v. If only one of the terms in the product which forms the integrand can be integrated easily, then there is no choice in the splitting. In some cases (as in the following example), a choice has to be made.

Example: Find $I(x) = \displaystyle\int xe^{-ax}\,dx$ where a is a constant.

Solution. The integrand is the product of x and e^{-ax}. These are both easy to differentiate or integrate. If we were to identify e^{-ax} with u and x with $\dfrac{dv}{dx}$, then application of Equation (1.5) would generate an integral that is no less difficult to evaluate than the original problem. However, if we let $u = x$ and $\dfrac{dv}{dx} = e^{-ax}$, then $\dfrac{du}{dx} = 1$ and $v = -\dfrac{1}{a}e^{-ax}$. Applying Equation (1.5), we get

$$\begin{aligned} I(x) &= \int xe^{-ax}\,dx = x\left(-\frac{1}{a}e^{-ax}\right) - \int\left(-\frac{1}{a}e^{-ax}\right)1\,dx \\ &= -\frac{x}{a}e^{-ax} + \frac{1}{a}\int e^{-ax}\,dx \\ &= -\frac{x}{a}e^{-ax} - \frac{1}{a^2}e^{-ax} + C. \end{aligned}$$

Note that if one of the terms in the product integrand is a polynomial, this will typically be the best choice for u, since differentiation will reduce the degree of the polynomial. It may sometimes be necessary to apply Equation (1.5) multiple times to get a final answer.

Example: Find $I(x) = \displaystyle\int e^{ax} \sin bx\, dx$, where a and b are constants. This type of integral is very important in the context of Fourier Series and Transforms.

Solution: In this case, both components of the product can be easily differentiated or integrated. Indeed, we can choose to identify either component with u. We choose to set

$$u = \sin bx \quad \text{and} \quad \frac{dv}{dx} = e^{ax} \quad \Rightarrow \quad \frac{du}{dx} = b\cos bx \quad \text{and} \quad v = \frac{1}{a}e^{ax}$$

Applying Equation (1.5), we get

$$\begin{aligned} I(x) &= \int e^{ax} \sin bx\, dx = \sin bx \left(\frac{1}{a}e^{ax}\right) - \int \left(\frac{1}{a}e^{ax}\right)(b\cos bx)\, dx \\ &= \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}\int e^{ax}\cos bx\, dx \\ &\equiv \frac{1}{a}e^{ax}\sin bx - \frac{b}{a}J(x). \end{aligned}$$

This does not appear at first to have got us very far, as the new integral $J(x)$ we have obtained is of a similar form to $I(x)$, and no easier to find... However, the important thing to remember is not to lose heart. We need to apply Equation (1.5) for a second time to $J(x)$. We need to choose

$$u = \cos bx \quad \text{and} \quad \frac{dv}{dx} = e^{ax} \quad \Rightarrow \quad \frac{du}{dx} = -b\sin bx \quad \text{and} \quad v = \frac{1}{a}e^{ax}.$$

Note that if we chose u and v the other way round, we would simply reverse the operation we carried out in the first application of Equation (1.5)! Applying Equation (1.5) to $J(x)$, we obtain

$$\begin{aligned} J(x) &= \int e^{ax} \cos bx\, dx = \cos bx \left(\frac{1}{a}e^{ax}\right) - \int \left(\frac{1}{a}e^{ax}\right)(-b\sin bx)\, dx \\ &= \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}\int e^{ax}\sin bx\, dx = \frac{1}{a}e^{ax}\cos bx + \frac{b}{a}I(x). \end{aligned}$$

If we now substitute this expression for $J(x)$ in to our earlier expression for $I(x)$, we obtain an equation for $I(x)$:

$$I(x) = \frac{1}{a}e^{ax}\sin bx - \frac{b}{a^2}\cos bx e^{ax} - \frac{b^2}{a^2}I(x).$$

We then rearrange this equation to bring the two terms involving $I(x)$ on to the left hand side:

$$(a^2 + b^2)I(x) = ae^{ax} \sin bx - b \cos bx e^{ax}.$$

And finally, we obtain the desired result:

$$I(x) = \frac{ae^{ax} \sin bx - b \cos bx e^{ax}}{a^2 + b^2} = e^{ax} \frac{a \sin bx - b \cos bx}{a^2 + b^2} + C,$$

where we have included the necessary constant of integration C at the final step.

Exercise: Obtain the same expression for $I(x)$ by setting $u = e^{ax}$ and $\dfrac{dv}{dx} = \sin bx$.

1.4 The Definite Integral

1.4.1 Definition of the Definite Integral

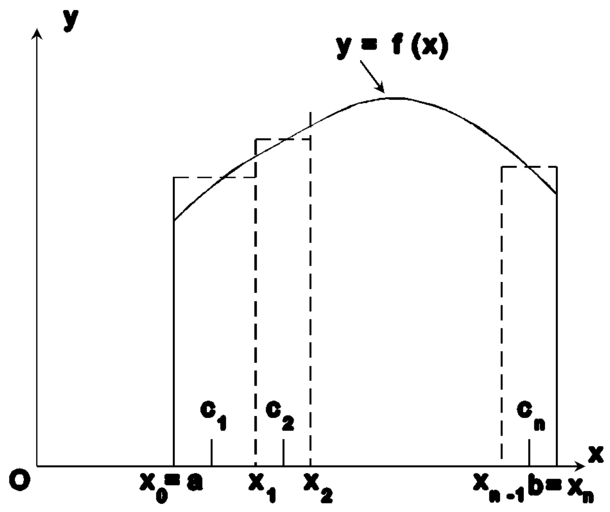


Figure 1.1: Definition of the definite integral

To define the definite integral consider Figure (1.1), which shows the function $y = f(x)$ defined over the interval $a \leq x \leq b$. Sub-divide the interval $a < x < b$ into n sub-intervals by the points $x_1, x_2, \ldots, x_n$ chosen arbitrarily.
In each of the intervals $(a, x_1), (x_1, x_2), \ldots, (x_{n-1}, b)$, choose points $c_1, c_2, \ldots, c_n$ arbitrarily.
Form the sum

$$S_n = f(c_1)(x_1 - a) + f(c_2)(x_2 - x_1) + \cdots + f(c_n)(b - x_{n-1}). \tag{1.6}$$

Looking at the diagram we can see that this is an approximation to the area under the curve $y = f(x) > 0$.
By writing $x_0 = a, \quad x_n = b$ and $\Delta_k = x_k - x_{k-1}$, this sum can be expressed as

$$S_n = \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta_k. \tag{1.7}$$

Now let the number of sub-divisions, n, increase to infinity in such a way that the maximum size of the intervals approaches zero. If the value of the sum in Eqn. (1.7) approaches a limiting value, which **does not** depend upon the manner in which the interval has been sub-divided, then this limit is denoted by

$$I = \lim_{n \to \infty} (S_n) = \int_a^b f(x)\,dx. \tag{1.8}$$

I is called the **definite integral** of f from a to b. From the diagram we can see that this integral is equal to the area under the curve $y = f(x) > 0$.

The function f is referred to as the **integrand** (as in the indefinite integral) and a and b are the **lower** and **upper limits**, respectively. The above formulation of the integral is called **the Riemann integral**. There are other (more technical) definitions of integration. An important result is that **any function which is continuous on an interval possess a Riemann integral on that same interval**. Another way of putting this is to say that continuous functions are **integrable**.

Note that although we have sketched the function f in Figure (1.1) to be such that $f(x) > 0$ for $a \leq x \leq b$, this is not necessary. If $f(x)$ is negative for some range of x, then the definite integral over this integral is also negative (one can think of areas below the x-axis as negative). This can provide a useful indication if you happen to make an error in your evaluation of a definite integral. If you know that $f(x) > 0$ over the range of x for which you are evaluating the integral, but you obtain a negative answer for the integral, then you know you have made a mistake somewhere along the line (probably a sign slip).

Although the area under a curve is a very useful interpretation of the definition of the definite integral, there are many other applications such as volumes of revolution, centres of mass and moments of inertia. The physical interpretation depends on what the integrand, in the above case $f(x)$, and the variable, x, represent for the physical problem in which it arises. For example if x is length along a wire and $f(x)$ represents the density (mass per unit length) of the material that makes up the wire at position x, the integral provides the mass of the wire between positions a and b.

Since a definite integral depends upon only upon the limits of integration, we can use **any** variable as the variable of integration. So, for example

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

For this reason, the variable of integration (either x or t in the above) is often called a **dummy variable**.

So although in these notes we typically use x as the integration variable, you will come across integrals that use different variable letters, sometimes chosen to reflect the nature of the integrand (e.g. x often refers to distance, t to time, etc.). Note also that when we refer to the area under the curve, this does not necessarily have the *dimensions* of area (i.e. [length]^2). For example, if $v(t)$ is the velocity of a particle along a line, then

$$D = \int_a^b v(t)\,dt$$

is the *distance* travelled by the particle between time $t = a$ and time $t = b$. While this is still the area under the curve, it clearly does not have the dimensions of physical area (it actually has the dimensions of [speed]$\times$[time] $=$ [distance]).

1.4.2 Properties of the Definite Integral

If $f(x)$ and $g(x)$ are each integrable on $[a, b]$, then

1. $$\int_a^a f(x)\,dx = 0,$$

2. $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

3. $$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx, \quad \text{(where } \alpha \text{ is any constant)}$$

4. $$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx,$$

5. $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad \text{for any } c\,\epsilon\,(a, b)$$

1.4.3 Definite integrals of even and odd functions

Even functions

Recall that $f(x)$ is an even function if $f(x) = f(-x)$. Now, for any function f

$$\int_{-a}^a f(x)\,dx = \int_{-a}^0 f(x)\,dx + \int_0^a f(x)\,dx \quad \text{(using Property 5 above with } c = 0\text{)}.$$

If we make the substitution $u = -x$ ($\Rightarrow du = -dx$) in the first integral, we see that

$$\begin{aligned}
\int_{-a}^a f(x)\,dx &= \int_a^0 -f(-u)\,du + \int_0^a f(x)\,dx \\
&= -\int_a^0 f(-u)\,du + \int_0^a f(x)\,dx \quad \text{(using Property 3 above with } \alpha = -1\text{)} \\
&= -\int_a^0 f(u)\,du + \int_0^a f(x)\,dx \quad \text{(using the fact that } f \text{ is even)} \\
&= \int_0^a f(u)\,du + \int_0^a f(x)\,dx \quad \text{(using Property 2 above)} \\
&= 2\int_0^a f(x)\,dx. \quad \text{(using Property 4 above, and dummy variable equivalence)}
\end{aligned}$$

Odd functions

Recall that $f(x)$ is an odd function if $f(x) = -f(-x)$. Using a very similar argument to that used for even functions above, we can show that if f is an **odd** function then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

For example, Figure (1.2) shows the integral of $\sin x$ over the range $[-\pi, \pi]$. The two shaded regions have the same area, but opposite signs, and so their algebraic sum is zero.

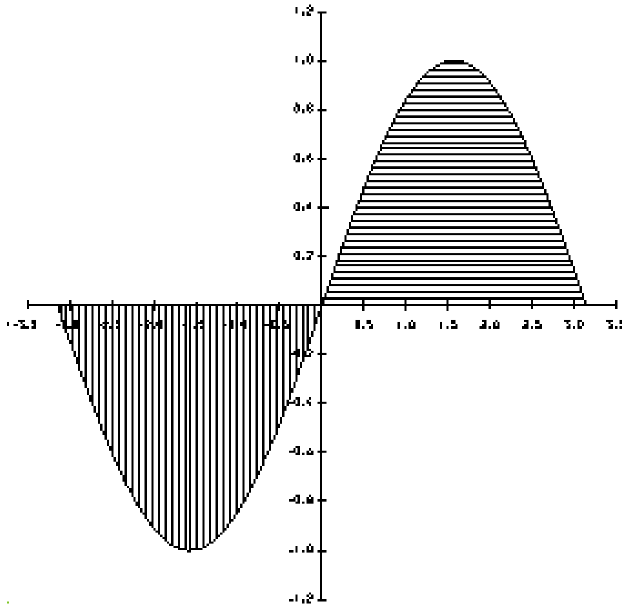


Figure 1.2: The integral of $\sin x$ between $x = -\pi$ and $x = \pi$.

1.4.4 Relation between Indefinite and Definite Integrals

As one might expect, there is a relationship between definite and indefinite integrals. If we can find an indefinite integral of an integrand, then the following result enables us to calculate definite integrals of the same integrand without direct use of the definition.

If $f(x)$ is continuous in $[a, b]$ and $F(x)$ is any function such that $dF/dx = f(x)$ (i.e. F is an indefinite integral of f) then

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a). \tag{1.9}$$

The following shorthand notation is often used

$$\int_{a}^{b} f(x)\,dx = F(b) - F(a) \equiv [F(x)]_a^b\,.$$

Note that we don't need to worry about a constant of integration in $F(x)$, since this will always cancel out when we calculate $F(b) - F(a)$.

Example: Evaluate $I = \int_1^{10} x\,dx$.

Solution: In this example, $f(x) = x$, and so an indefinite integral is $F(x) = \dfrac{x^2}{2}$ (remember: no need to include a constant of integration). So

$$I = \int_1^{10} x\,dx = \left[\frac{x^2}{2}\right]_1^{10} = \frac{10^2}{2} - \frac{1^2}{2} = 50 - \frac{1}{2} = 49.5.$$

In this case, it is easy to calculate the area under the curve $y = x$ directly, and to verify that the result we have obtained is correct.

Exercise: Evaluate $I = \int_0^{\pi} \sin x\,dx$.

Solution: In this example, $f(x) = \sin x$, and so an indefinite integral is $F(x) = -\cos x$. Therefore

$$I = \int_0^{\pi} \sin x\,dx = [-\cos x]_0^{\pi} = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

Example: Evaluate $I = \int_0^{2} \dfrac{3x-2}{(x+1)(x^2+4)}\,dx$.

Solution: We use partial fractions to cast the integrand into a combination of standard forms. Set

$$\frac{3x-2}{(x+1)(x^2+4)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+4}.$$

Then $3x - 2 \equiv A(x^2+4) + (Bx+C)(x+1)$.

Setting $\quad x = -1 \Rightarrow -5 = 5A \Rightarrow \underline{A = -1}$.

We now compare the coefficients of the terms in x^2 and x:

$$\begin{aligned}
\text{Coefficient of } x^2 : \quad & 0 = A + B \\
\Rightarrow \quad & B = -A \Rightarrow \underline{B = 1} \\[1em]
\text{Coefficient of } x : \quad & 3 = B + C \\
\Rightarrow \quad & C = 3 - B \Rightarrow \underline{C = 2}
\end{aligned}$$

Thus we have

$$
\begin{aligned}
I = \int_0^2 \frac{3x-1}{(x+1)(x^2+4)}\,dx &= -\int_0^2 \frac{dx}{x+1} + \int_0^2 \frac{x+2}{x^2+4}\,dx \\
&= -\left[\ln(x+1)\right]_0^2 + \int_0^2 \frac{x}{x^2+4}\,dx + 2\int_0^2 \frac{dx}{x^2+4} \\
&= -(\ln 3 - \ln 1) + \frac{1}{2}\left[\ln\left(x^2+4\right)\right]_0^2 + \frac{2}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2 \\
&= -\ln 3 + \frac{1}{2}(\ln 8 - \ln 4) + \left(\tan^{-1} 1 - \tan^{-1} 0\right) \\
&= -\ln 3 + \frac{1}{2}\ln\left(\frac{8}{4}\right) + \frac{\pi}{4} \\
&= -\frac{1}{2}\ln 9 + \frac{1}{2}\ln 2 + \frac{\pi}{4} \\
&= \frac{1}{2}\ln\frac{2}{9} + \frac{\pi}{4}.
\end{aligned}
$$

1.4.5 Using substitution to evaluate definite integrals

Sometimes, we may need to use substitution to find the indefinite integral (as in the examples and exercises in the previous section). If we make a substitution $x \to u = g(x)$, then when we are evaluating a definite integral, we also need to change the limits of the integration. Specifically, if the lower and upper limits of the integral are $x = a$ and $x = b$, respectively, then after we have applied the substitution $x \to u = g(x)$, the new limits are $u = g(a)$ and $u = g(b)$, respectively.

Example: Evaluate $I = \int_1^3 4x^3\,dx$ using the substitution $u = x^2$.

Solution: Note first that we could of course evaluate this integral without using a substitution. However, if $u = x^2$, then $du = 2x\,dx$ and $x = 1 \Rightarrow u = 1$ and $x = 3 \Rightarrow u = 9$. Therefore

$$I = \int_1^3 4x^3\,dx = \int_1^9 2u\,du = \left[u^2\right]_1^9 = 81 - 1 = 80.$$

Exercise: Evaluate $I = \int_1^3 4x^3\,dx$ directly to verify this result.

Example: Evaluate $I = \int\limits_0^{\tan^{-1}(2)} \frac{\sec^2 x}{4+\tan^2 x}\, dx.$

Solution: The integrand is of the form $f[g(x)]g'(x)$ where

$$g(x) = \tan x \quad \text{and} \quad f(x) = \frac{1}{4+x^2}$$

$$\Rightarrow \ f[g(x)] = \frac{1}{4+[g(x)]^2} = \frac{1}{4+\tan^2 x}.$$

Set $u = g(x) = \tan x \ \Rightarrow \ du = \sec^2 x \ dx$ and change the limits from x to u:

$$x = 0 \ \Rightarrow \ u = \tan(0) = 0$$

$$x = \tan^{-1}(2) \ \Rightarrow \ u = \tan[\tan^{-1}(2)] = 2.$$

Hence

$$\begin{aligned} I = \int\limits_0^{\tan^{-1}(2)} \frac{\sec^2 x}{4+\tan^2 x}\, dx &= \int\limits_0^2 \frac{du}{4+u^2} \qquad\qquad (u = \tan x) \\ &= \frac{1}{2}\left[\tan^{-1}\left(\frac{u}{2}\right)\right]_0^2 \\ &= \frac{1}{2}\left(\tan^{-1} 1 - \tan^{-1} 0\right) \\ &= \frac{1}{2}.\frac{\pi}{4} \\ &= \frac{\pi}{8}. \end{aligned}$$

1.5 Improper integrals

One type of improper integral occurs when one or both of the limits of integration of a definite integral tend to $\pm\infty$ (i.e. the range of integration is infinite). In this case the following method is used to define the indefinite integral:

Let $F(X) = \int\limits_a^X f(x)\, dx$ and let $F(X)$ tend to a limit as $X \to \infty$. Then we define

$$\int\limits_a^\infty f(x)\, dx = \lim_{X\to\infty} \int\limits_a^X f(x)\, dx.$$

If this limit does not exist then no meaning is attached to $\displaystyle\int_a^\infty f(x)\,dx$.

Similarly

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{Y\to\infty} \int_{-Y}^{b} f(x)\,dx,$$

if the limit exists.

Also

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{X\to\infty} \lim_{Y\to\infty} \int_{-Y}^{X} f(x)\,dx,$$

where the two limiting processes must be done separately.

Example: Evaluate $\displaystyle\int_0^\infty \frac{dx}{x^2+4}$.

Solution:

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^2+4} &= \lim_{X\to\infty} \int_0^X \frac{dx}{x^2+4} \\
&= \lim_{X\to\infty} \frac{1}{2}\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^X \\
&= \lim_{X\to\infty} \frac{1}{2}\tan^{-1}\left(\frac{X}{2}\right) \qquad [\text{since}\tan^{-1}(0)=0] \\
&= \frac{1}{2}.\frac{\pi}{2} = \frac{\pi}{4}.
\end{aligned}$$

It is conventional to perform the limiting operation implicitly, using the notation

$$\left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^\infty = \lim_{X\to\infty} \left[\tan^{-1}\left(\frac{x}{2}\right)\right]_0^X.$$

The term "improper" is also used when $f(x)$ has one (or more) points of discontinuity in the range of integration. This is treated as follows:

If $f(x)$ is discontinuous at $x=a$ but continuous in $(a,b]$, then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon\to 0_+} \int_{a+\epsilon}^{b} f(x)\,dx,$$

provided, of course, that the limit exists.

If the point of discontinuity is within the range of integration (at c, say, where $a < c < b$) then we define

$$\int_a^b f(x)\,dx = \lim_{\epsilon \to 0+} \int_a^{c-\epsilon} f(x)\,dx \;+\; \lim_{\eta \to 0+} \int_{c+\eta}^{b} f(x)\,dx,$$

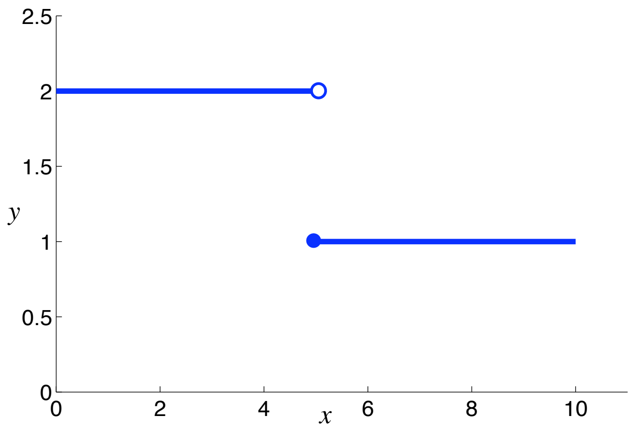
provided that **both** of the limits exist.

Again it is conventional to not write the limiting operation explicitly. The above can be written

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example: Evaluate $\displaystyle\int_0^{10} f(x)\,dx$, where $f(x) = \begin{cases} 2 & (x < 5) \\ 1 & (x \geq 5) \end{cases}$

Solution:



The graph of $y = f(x)$ has the form shown in the figure. Noting that it has a discontinuity at $x = 5$, we write

$$\begin{aligned}
\int_0^{10} f(x)\,dx &= \int_0^5 f(x)\,dx + \int_5^{10} f(x)\,dx \\
&= \int_0^5 2\,dx + \int_5^{10} 1\,dx \\
&= [\,2x\,]_0^5 + [\,x\,]_5^{10} \\
&= (10-0) + (10-5) = 15.
\end{aligned}$$

Continuous functions also sometimes need to be subdivided (for example, if they are not differentiable everywhere in the range of integration).

Example: Evaluate $\displaystyle\int_{-1}^{2} |x|\, dx$.

Solution. First we need to look carefully at the integrand $|x|$. It is defined as follows:

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

so that $|x|$ is a continuous function which is not differentiable (see Figure (1.3)). It is differentiable for $x > 0$ and also for $x < 0$, but not over the full range of x i.e. for $x \in \mathbb{R}$.

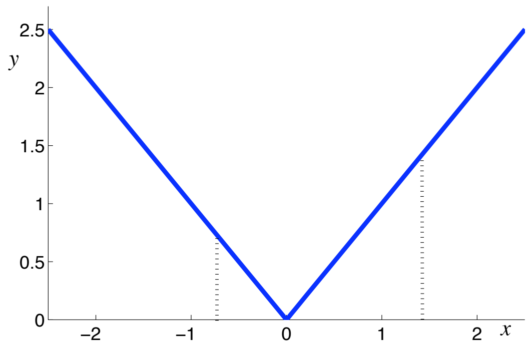


Figure 1.3: The graph of $y = f(x) = |x|$.

Since $x = 0$ is the difficult point, we split the integral into two parts as follows:

$$\begin{aligned}
\int_{-1}^{2} |x|\, dx &= \int_{-1}^{0} |x|\, dx + \int_{0}^{2} |x|\, dx \\
&= \int_{-1}^{0} (-x)\, dx + \int_{0}^{2} x\, dx \\
&= \left[-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} \\
&= 0 - \left[-\frac{1}{2}(-1)^2\right] + \frac{1}{2}(2)^2 - 0 \\
&= \frac{1}{2} + \frac{4}{2} = \frac{5}{2}.
\end{aligned}$$

2 Matrices and Linear Equations

2.1 Matrices

A matrix is a rectangular array of numbers. For example,

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 5 & 7 \\ -3 & 2 & 1 & 5 \end{bmatrix}$$

is a matrix. It has 3 rows and 4 columns and is called a 3 x 4 matrix.
A matrix with m rows and n columns is called an m x n matrix. It has mn entries.
Sometimes round brackets are used in place of [ ] for matrices. For example, some text books would use

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 5 & 7 \\ -3 & 2 & 1 & 5 \end{pmatrix}$$

for the above matrix.

Important Notes:

1. Be careful to use [ ] or ( ) for matrices. Vertical lines such as | | are used for **determinants** (see later).

2. **Do not** confuse matrices and determinants.

3. Matrices are rectangular arrays of numbers, they do NOT have a numerical value.

2.1.1 Addition of matrices.

Two matrices can be added if and only if they are the same size. i.e. If A is an $m \times n$ matrix then $A + B$ exists if and only if B is also an $m \times n$ matrix. In this cases $A + B$ is obtained by adding corresponding elements.

Example: Find $A + B$, if it exists, for the following matrices.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & -1 \\ -1 & 2 & 7 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 10 & 11 \\ 0 & 5 & -3 & 1 \\ 3 & 7 & -6 & 7 \end{bmatrix}.$$

Let $C = \begin{bmatrix} 0 & 3 & 1 \\ 1 & -1 & 0 \\ 5 & 2 & 7 \end{bmatrix}$. Does $A + C$ exist?

Solution: $A + B$ exists because both are 3×4 matrices

$$A+B=\begin{bmatrix} 1-1 & 2+3 & 3+10 & 4+11 \\ 0+0 & 1+5 & 5-3 & -1+1 \\ -1+3 & 2+7 & 7-6 & 6+7 \end{bmatrix}$$

$$A+B=\begin{bmatrix} 0 & 5 & 13 & 15 \\ 0 & 6 & 2 & 0 \\ 2 & 9 & 1 & 13 \end{bmatrix}$$

$A+C$ does not exist because A is a 3×4 matrix, C is a 3×3 matrix so A and C are not the same size.

2.1.2 Equality of Matrices

Two matrices are equal if and only if

1. the matrices are the same size, and
2. the corresponding entries in the two matrices are the same.

Example: Let $A=\begin{bmatrix} 0 & 1 & 6 & 5 \\ -1 & 2 & 3 & 4 \\ 1 & 7 & -1 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} a & b & c & d \\ f & g & h & l \\ m & n & p & q \end{bmatrix}.$

Then $A=B \Longleftrightarrow a=0, \quad b=1, \quad c=6, \quad d=5$

$f=-1, \quad g=2, \quad h=3, \quad l=4, \quad m=1$

$n=7, \quad p=-1 \quad q=0.$

2.1.3 Scalar multiples of a Matrix

Let A be a matrix and λ be a scalar, then λA is obtained from A by multiplying every element in A by λ.

Example: Let

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad B=\begin{bmatrix} -1 & 0 & 4 \\ 2 & 1 & 7 \end{bmatrix}.$$

Find $A+2B$ and $A-2B$.

Solution:

$$2B=\begin{bmatrix} -2 & 0 & 8 \\ 4 & 2 & 14 \end{bmatrix}$$

$$\therefore \quad A + 2B = \begin{bmatrix} -1 & 2 & 11 \\ 8 & 7 & 20 \end{bmatrix}$$

By $A - 2B$ we mean $A + (-2)B$, and so

$$A - 2B = \begin{bmatrix} 3 & 2 & -5 \\ 0 & 3 & -8 \end{bmatrix}$$

Notation: Given an $m \times n$ matrix A we often use a_{rs} for the element in row r and column s and write $A = [a_{rs}]$ or $A = [a_{rs}]_{m\times n}$. With this notation the elements of row 2 are

$$a_{21} \quad a_{22} \quad a_{23} \cdots a_{2n}.$$

The elements of column 3 are

$$\begin{matrix} a_{13} \\ a_{23} \\ a_{33} \\ \vdots \\ a_{m3} \end{matrix}$$

So

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

2.1.4 Transpose of a Matrix

Given a matrix A its **transpose** (denoted by A^T) is obtained by interchanging the rows and columns of A.

For example if $A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \\ 3 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

If $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}$ then $B^T = \begin{bmatrix} b_{11} & \cdots & b_{m1} \\ b_{12} & \cdots & b_{m2} \\ \vdots & & \vdots \\ b_{1n} & \cdots & b_{mn} \end{bmatrix}$.

2.1.5 Symmetric Matrix

A matrix B is said to be **symmetric** if $B = B^T$. Thus a matrix B is symmetric if

1. B is a square matrix, *and*

2. $b_{rs} = b_{sr}$.

For example, the matrix

$$B = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 3 & -1 \\ 7 & -1 & 8 \end{bmatrix}$$

is symmetric, since it is unchanged when rows and columns are interchanged.

2.1.6 Skew-symmetric (anti-symmetric) matrix

A matrix B is **skew-symmetric** (also called **anti-symmetric**) if

$$B^T = -B.$$

Thus a matrix B is skew-symmetric if

1. B is a square matrix, *and*

2. $b_{rs} = -b_{sr}$.

In particular $r = s$ gives $b_{rr} = -b_{rr}$ i.e. $b_{rr} = 0$ and the *diagonal elements* b_{rr} are all zero.

For example $B = \begin{bmatrix} 0 & 1 & 3 & -5 \\ -1 & 0 & -2 & -4 \\ -3 & 2 & 0 & -\frac{1}{2} \\ 5 & 4 & \frac{1}{2} & 0 \end{bmatrix}$ is skew-symmetric.

2.1.7 Identity (Unit) Matrix

The $n \times n$ matrix with 1 at each place on the leading diagonal and zeros everywhere else is called the **Identity Matrix** (or **Unit Matrix**) I_n. Thus

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

If we don't need to emphasise the size of the matrix, we can just write I in place of I_n.

Later, we will see that this identity matrix plays the same role in matrix multiplication as the number 1 does in ordinary multiplication of real numbers.

2.1.8 Diagonal Matrix

An $n \times n$ matrix with at least one non-zero number on the leading diagonal and zeros everywhere else is call a *diagonal matrix*.
For example

$$A = \begin{bmatrix} 4 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 6 \end{bmatrix}$$

is a diagonal matrix.

2.1.9 Matrix Multiplication

We do not define matrix multiplication in what might appear to be the obvious way (i.e. multiplying corresponding elements) as this would not be at all useful.

To find a way which would make matrices useful let us consider three matrices:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Note: Row and column vectors are just special examples of matrices.

Now let us look at a system of linear equations involving the elements of these matrices.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

It would be convenient to be able to write this as a single matrix equation:

$$AX = B$$

This will be true if we use a 'row by column' definition for matrix multiplication.
Let A be an $m \times n$ matrix (m rows, n columns) and let B be a $n \times p$ matrix (n rows, p columns). Then the matrix product AB exists **precisely because** the number of columns in A equals the number of rows in B. In this case, the matrix AB is an $m \times p$ matrix.

Important Note: The order is important. Do not swap the order. As you will see later AB and BA are, in general, **not** equal. In other words, matrix multiplication **does not** commute.

In general, we multiply matrices in the following way:
Let $C = AB$, where A is $m \times n$ and B is $n \times p$. The element c_{rs} from row r and column s of C is obtained by taking row r of A and column s of B. We then multiply corresponding elements and add (i.e. we calculate the *scalar product* of the two vectors). Because the number of elements in a row of A equals the number of elements in a column of B, we can pair them together to make this formula work. This is why we talk about 'row into column' multiplication.

Example: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}.$

Then A is 2×3 and B is 3×2, and we can calculate the 2×2 matrix AB:

$$AB = \begin{bmatrix} row1 * col1 & row1 * col2 \\ row2 * col1 & row2 * col2 \end{bmatrix}$$

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}.$$

Note that we can also form the product BA, but this is a 3×3 matrix and so AB and BA are different sizes in this case and so are clearly **not** equal.

Example: If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$, find AB and BA if they exist.

Solution: A is 2×3 and B is 3×3 so AB exists and is 2×3.

$$AB = \begin{bmatrix} 1 \times 1 + (-1) \times 0 + 2 \times 1 & 1 \times 2 + (-1) \times (-1) + 2 \times 2 & 2 \\ 3 \times 1 + 0 \times 0 + 1 \times 1 & 3 \times 2 + 0 \times (-1) + 2 \times 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 2 \\ 4 & 8 & 1 \end{bmatrix}.$$

The product BA does not exist, since the number of columns of B (3) does not equal the number of rows of A (2).

Exercise: For the matrices

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 5 & -1 \\ 3 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix},$$

decide whether the following exist: (a) AB, (b) AC, (c) CB. Find those which do exist.

Solution:

A is 2×3, B is 3×2 and C is 2×2. So AB is $(2 \times 3).(3 \times 2) \Rightarrow 2 \times 2$, and so AB exists.
AC is $(2 \times 3).(2 \times 2)$ and $3 \neq 2 \Rightarrow AC$ does not exist.
CB is $(2 \times 2).(3 \times 2)$ and $2 \neq 3 \Rightarrow CB$ does not exist.

$$AB = \begin{bmatrix} 3 \times 2 + 1 \times 5 + 1 \times 3 & 3 \times 1 + 1 \times (-1) + 1 \times 1 \\ 2 \times 2 + (-1) \times 5 + 0 \times 3 & 2 \times 1 + (-1) \times (-1) + 0 \times 1 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -1 & 3 \end{bmatrix}.$$

Notes:

1. Even when AB and BA both exist, in general $AB \neq BA$.

   e.g. let $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$.

   Then $AB = \begin{bmatrix} 8 & 4 \\ -4 & 1 \end{bmatrix}$ and $BA = \begin{bmatrix} 7 & 5 \\ -2 & 2 \end{bmatrix}$.

2. Provided the products exist, then

$$\begin{aligned} (AB)C &= A(BC), \\ (A+B)C &= AC + BC. \end{aligned}$$

3. If the product AB exists, then $(AB)^T = B^T A^T$

   and more generally $(ABC \ldots XY)^T = Y^T X^T \ldots C^T B^T A^T$.

4. If $AB = 0$ (where 0 is the **zero matrix**, *all* of whose entries are zero), then this does **not** necessarily mean that $A = 0$ or $B = 0$ (so we have lost a cancellation law).

   e.g. let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

5. Assuming AB and AC both exist, then the relation $AB = AC$ does **not** necessarily mean that $B = C$.

   e.g. let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

   Then $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and so $AB = AC$, but here $B \neq C$.

2.2 Determinants

A **determinant** is a single number associated with a square matrix (i.e. a matrix that has an equal number of rows and columns). The determinant of a square matrix A is denoted by $|A|$ or $det(A)$. If the matrix A is $n \times n$, then its determinant is said to be of **order** n.
Determinants are intimately connected with the solution of systems of linear equations and linear transformation of vectors. We will see some applications later in this section.

Determinants are not defined for non-square matrices.

Examples:
The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For example

$$\begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} = 3 \times 2 - 4 \times (-1) = 6 + 4 = 10.$$

The determinant of a 3×3 matrix can be evaluated as follows:

$$\begin{aligned} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \\ &= a_1 \left(b_2c_3 - b_3c_2\right) - a_2 \left(b_1c_3 - b_3c_1\right) + a_3 \left(b_1c_2 - b_2c_1\right). \end{aligned} \tag{2.1}$$

Note the alternate '+' and '–' signs.

So, for example,

$$\begin{aligned} \begin{vmatrix} 3 & 2 & -1 \\ 5 & 1 & 2 \\ -2 & 4 & -5 \end{vmatrix} &= 3 \begin{vmatrix} 1 & 2 \\ 4 & -5 \end{vmatrix} - 2 \begin{vmatrix} 5 & 2 \\ -2 & -5 \end{vmatrix} + (-1) \begin{vmatrix} 5 & 1 \\ -2 & 4 \end{vmatrix} \\ &= 3(-5-8) - 2(-25+4) - 1(20+2) \\ &= -39 + 42 - 22 \\ &= -19. \end{aligned}$$

This expansion procedure can be generalised to higher order determinants. For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \\ d_2 & d_3 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \\ d_1 & d_3 & d_4 \end{vmatrix}$$

$$+a_3 \begin{vmatrix} b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \\ d_1 & d_2 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix}.$$

2.2.1 Minors and Cofactors

The **minor** associated with the element a_{ij} in the matrix A is defined as the determinant of the matrix obtained from A by deleting the row and column that contain the element a_{ij} (i.e. row i and column j).

For example, if A is the 3×3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then

$$\alpha_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}, \quad \alpha_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}a_{23} - a_{13}a_{21}, \text{ etc.}$$

The **cofactor** of the element a_{ij} is denoted by Aij and its value is

$$A_{ij} = (-1)^{i+j}\alpha_{ij}\,.$$

Thus $A_{11} = (-1)^{1+1}\alpha_{11} = a_{22}a_{33} - a_{23}a_{32}$,

and $A_{32} = (-1)^{3+2}\alpha_{32} = -(a_{11}a_{23} - a_{13}a_{21})$.

2.2.2 Expansion of a determinant

Recalling the expression (2.1) for the determinant of A, we see that

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\alpha_{11} - a_{12}\alpha_{12} + a_{13}\alpha_{13}.$$

We can thus evaluate the determinant of order 3 by expanding in terms of minors.

Noting that

$$\left.\begin{array}{rclcrcr} A_{11} & = & (-1)^{1+1} & \alpha_{11} & = & \alpha_{11} \\ A_{12} & = & (-1)^{1+2} & \alpha_{12} & = & -\alpha_{12} \\ A_{13} & = & (-1)^{1+3} & \alpha_{13} & = & \alpha_{13} \end{array}\right\} \quad (\dagger)$$

we can write

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13},$$

using (†). Note that the alternating signs that appear in the expansion by minors are absorbed into the cofactors, A_{ij}. This expression provides an expansion of $|A|$ by row 1.

We can equally well expand by row 2 or by row 3:

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} \quad \text{(row 2)}$$

or

$$|A| = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33} \quad \text{(row 3).}$$

Similarly we can expand a determinant by any given column. For example, expanding by column 2 gives

$$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

Exercise: Show that all the expressions for $|A|$ given above give the same answer for the matrix

$$A = \left[\begin{array}{rrr} 2 & 1 & -2 \\ -4 & 3 & 1 \\ 1 & -2 & 2 \end{array}\right].$$

General Rule: Generally, the determinant of a square matrix A can be evaluated by taking *any* row (or column) of A, multiplying each element of the row (or column) by its own cofactor, and summing the results.

Thus, if A is an $n \times n$ matrix, then for any $1 \leq i \leq n$:

$$|A| = \sum_{j=1}^{n} a_{ij}A_{ij} \quad \text{(expansion by row } i)$$

or

$$|A| = \sum_{j=1}^{n} a_{ji}A_{ji} \quad \text{(expansion by column } i).$$

2.2.3 The 'Alien' Cofactor Rule

For a 3×3 matrix A, it is easy to demonstrate that

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$$

and

$$a_{11}A_{12} + a_{21}A_{22} + a_{31}A_{32} = 0$$

These are examples of a general rule — the *'alien' cofactor rule*:

> The sum of the products of the elements of any row (or column) of a determinant with the cofactors of another row (or column) is always zero.

$$\text{i.e.} \quad \sum_{k=1}^{n} a_{ik}A_{jk} = 0 \quad \text{and} \quad \sum_{k=1}^{n} a_{ki}A_{kj} = 0 \quad \text{if } i \neq j.$$

Example: Expand $|A| = \begin{vmatrix} 3 & 1 & 2 \\ 2 & 4 & -1 \\ 1 & 1 & 2 \end{vmatrix}$ by column 3 and verify that

$$\begin{aligned} a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33} &= 0, \\ a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} &= 0. \end{aligned}$$

Solution:

$$\begin{aligned} A_{13} &= (-1)^{1+3}\begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} &= 2 - 4 &= -2 \\ A_{23} &= (-1)^{2+3}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} &= -(3 - 1) &= -2 \\ A_{33} &= (-1)^{3+3}\begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} &= 12 - 2 &= 10 \end{aligned}$$

Therefore

$$\begin{aligned} |A| &= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33} \\ &= 2 \times (-2) + (-1) \times (-2) + 2 \times 10 \ = \ 18. \end{aligned}$$

Also

$$a_{12}A_{13} + a_{22}A_{23} + a_{32}A_{33} = 1 \times (-2) + 4 \times (-2) + 1 \times 10 = 0$$

and

$$a_{11}A_{13} + a_{21}A_{23} + a_{31}A_{33} = 3 \times (-2) + 2 \times (-2) + 1 \times 10 = 0.$$

2.2.4 Properties of Determinants

The following properties hold for determinants of any order, but here we shall give examples for determinants of order 3.

1. The value of a determinant does **not** change if we take the transpose. So if A is a square matrix

   $$|A| = |A^T|$$

   $$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}.$$

2. If two rows (or two columns) are interchanged, the **sign** of the determinant changes (i.e. the value is multiplied by -1). Thus if we exchange rows 1 and 3:

   $$\begin{vmatrix} 3 & 4 & -1 \\ 2 & 4 & -7 \\ 0 & 3 & 0 \end{vmatrix} = -\begin{vmatrix} 0 & 3 & 0 \\ 2 & 4 & -7 \\ 3 & 4 & -1 \end{vmatrix} = -(-3)\begin{vmatrix} 2 & -7 \\ 3 & -1 \end{vmatrix} = 3(-2+21) = 57.$$

3. If two rows (or columns) are identical, the value of the determinant is zero. This is a consequence of the previous property: Interchange the two identical rows. This multiplies the value of the determinant by -1. But the new determinant is the same as the old one so $|A| = -|A| \Rightarrow |A| = 0$.

4. If A and B are both $n \times n$ matrices, then

   $$|AB| = |A||B|.$$

5. To multiply a determinant by the constant λ, we multiply all the elements of **one row** (or **one column**) by λ.

   For example:

   $$\lambda\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} \lambda a & \lambda b & \lambda c \\ d & e & f \\ g & h & i \end{vmatrix}.$$

   Contrast this with a scalar multiple of a matrix

   $$\lambda\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \lambda a & \lambda b & \lambda c \\ \lambda d & \lambda e & \lambda f \\ \lambda g & \lambda h & \lambda i \end{bmatrix}.$$

6. If we add a multiple of the elements of any one row (or column) to the corresponding elements of another row (or column), the value of the determinant does **not** change.

   For example

$$\begin{vmatrix} 6 & 2 & 4 \\ 4 & 4 & -1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 \\ 4 & 4 & -1 \\ 1 & 1 & 2 \end{vmatrix}. \qquad ([row1] - 2 \times [row3])$$

Example: Evaluate

$$|A| = \begin{vmatrix} 1 & 2 & -2 & -1 \\ -3 & -7 & 10 & 6 \\ 2 & 1 & -3 & 2 \\ 5 & 12 & -6 & -3 \end{vmatrix}.$$

Solution: We can evaluate this directly by expansion, or we can be smarter and first carry out some row/column operations using property 6 to simplify the determinant as we go along.

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & -1 & 4 & 3 \\ 2 & -3 & 1 & 4 \\ 5 & 2 & 4 & 2 \end{vmatrix} && \begin{aligned} \text{new column 2} &= \text{column } 2 - 2 \times (\text{column } 1) \\ \text{new column 3} &= \text{column } 3 + 2 \times (\text{column } 1) \\ \text{new column 4} &= \text{column } 4 + \text{column } 1 \end{aligned} \\
&= 1 \times \begin{vmatrix} -1 & 4 & 3 \\ -3 & 1 & 4 \\ 2 & 4 & 2 \end{vmatrix} && (\text{expanding by row 1}) \\
&= \begin{vmatrix} -1 & 0 & 0 \\ -3 & -11 & -5 \\ 2 & 12 & 8 \end{vmatrix} && \begin{aligned} \text{new column 2} &= \text{column } 2 + 4 \times (\text{column } 1) \\ \text{new column 3} &= \text{column } 3 + 3 \times (\text{column } 1) \end{aligned} \\
&= -1 \times \begin{vmatrix} -11 & -5 \\ 12 & 8 \end{vmatrix} && (\text{expanding by row 1}) \\
&= -1 \times [-11 \times 8 - (-5) \times 12] \ = -1 \times [-88 + 60] = 28
\end{aligned}$$

Exercise: Evaluate the determinant $|A| = \begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}$.

Solution:

$$
\begin{aligned}
|A| &= \begin{vmatrix} a & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 1 & 0 & c \end{vmatrix} \quad \begin{array}{lcl} \text{new column 1} & = & \text{column 1} - \text{column 2} \\ \text{new column 3} & = & \text{column 3} - \text{column 2} \\ \text{new column 4} & = & \text{column 4} - \text{column 2} \end{array} \\
&= a \begin{vmatrix} 1 & 0 & 0 \\ 1 & b & 0 \\ 1 & 0 & c \end{vmatrix} \quad \text{expanding by column 1} \\
&= a \begin{vmatrix} b & 0 \\ 0 & c \end{vmatrix} \quad \text{expanding by row 1} \\
&= abc
\end{aligned}
$$

Example: Simplify and evaluate $|A| = \begin{vmatrix} x & y & z \\ 1-2x & 2-2y & 3-2z \\ 2 & 3 & 4 \end{vmatrix}$.

Solution:

$$
\begin{aligned}
|A| &= \begin{vmatrix} x & y & z \\ 1-2x & 2-2y & 3-2z \\ 2 & 3 & 4 \end{vmatrix} \\
&= \begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{vmatrix} \quad \text{new row 2} = \text{row 2} + 2 \times (\text{row 1}) \\
&= \begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} \quad \text{new row 3} = \text{row 3} - \text{row 2} \\
&= x \begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} + z \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} \quad \text{expanding by row 1} \\
&= -x + 2y - z.
\end{aligned}
$$

2.3 The inverse of a square matrix

If a is a scalar, its inverse is $a^{-1} = 1/a$ and $a * a^{-1} = 1 = a^{-1} * a$ provided that $a \neq 0$.

Suppose A is a square $n \times n$ matrix. If we can find a square matrix B (also $n \times n$) such that $AB = I$ then also $BA = I$ and B is called the **inverse** of A and B is denoted by A^{-1}.

2.3.1 Existence of the inverse and singular matrices

We saw in the previous section that if A and B are $n \times n$ matrices, then

$$|AB| = |A||B|.$$

So if there exists a matrix $B = A^{-1}$ such that $AB = I$, then

$$|AB| = |A||B| = |I| = 1 \quad \text{(since } |I| = 1 \text{ for all } n\text{).}$$

Since this statement can only be true if $|A| \neq 0$, the inverse of a square matrix A exists if and only if $|A| \neq 0$.

A square matrix A is said to be **singular** if $|A| = 0$. If $|A| \neq 0$ then A is **non-singular**.

Example: Determine whether $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ has an inverse.

Solution: $|A| = 1 \times 6 - 2 \times 3 = 0$, and so A is singular and has no inverse.

2.3.2 The inverse of a 2×2 matrix

Let A be the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

If $|A| = ad - bc \neq 0$, then the inverse of A is given by

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

To demonstrate that this result is true, let

$$B = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then

$$
\begin{aligned}
AB &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\
&= \frac{1}{|A|} \begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -cb+ad \end{bmatrix} \\
&= \frac{1}{|A|} \begin{bmatrix} ad-bc & 0 \\ 0 & -cb+ad \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad (\text{since } |A| = ad-bc).
\end{aligned}
$$

Similarly $BA = I$ (Exercise). Therefore $B = A^{-1}$ as claimed.

2.3.3 The inverse of a 3 x 3 matrix

Let A be the 3×3 non-singular matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.
$$

Let C be the *transpose* of the matrix of cofactors of A:

$$
C = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T.
$$

We call C the **adjoint** of A (or the **adjugate** of A) and we write $C = adj A$.

The inverse of A is

$$
A^{-1} = \frac{1}{|A|} adj A.
$$

As for the case of a 2×2 matrix above, we can demonstrate the truth of this statement by direct calculation.

Let

$$
B = \frac{1}{|A|} adj A.
$$

Then

$$
\begin{aligned}
AB &= \frac{1}{|A|}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \\
&= \frac{1}{|A|}\begin{bmatrix} \sum_{j=1}^{3} a_{1j}A_{1j} & \sum_{j=1}^{3} a_{1j}A_{2j} & \sum_{j=1}^{3} a_{1j}A_{3j} \\ \sum_{j=1}^{3} a_{2j}A_{1j} & \sum_{j=1}^{3} a_{2j}A_{2j} & \sum_{j=1}^{3} a_{2j}A_{3j} \\ \sum_{j=1}^{3} a_{3j}A_{1j} & \sum_{j=1}^{3} a_{3j}A_{2j} & \sum_{j=1}^{3} a_{3j}A_{3j} \end{bmatrix} \\
&= \frac{1}{|A|}\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} \qquad \text{using the `Alien' Cofactor rule to get zeros} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I
\end{aligned}
$$

Similarly $BA = I$ (Exercise). Therefore we have demonstrated that $B = A^{-1}$.

Note that the result we stated for the inverse of a 2×2 matrix is just a special case of this result using the adjoint matrix. Also note that the reasoning used to demonstrate the result for a 3×3 matrix extends to any $n \times n$ matrix.

Example: Find A^{-1} when $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 3 \\ 1 & 4 & -1 \end{bmatrix}$.

Solution: First find the cofactors:

$$
\begin{aligned}
A_{11} &= (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 4 & -1 \end{vmatrix} = -11 \\
A_{12} &= (-1)^{1+2}\begin{vmatrix} 0 & 3 \\ 1 & -1 \end{vmatrix} = 3 \\
A_{13} &= (-1)^{1+3}\begin{vmatrix} 0 & -1 \\ 1 & 4 \end{vmatrix} = 1 \\
A_{21} &= (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} = 5 \\
A_{22} &= (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3
\end{aligned}
$$

$$
\begin{aligned}
A_{23} &= (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} &= -7 \\
A_{31} &= (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} &= 4 \\
A_{32} &= (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ 0 & 3 \end{vmatrix} &= -6 \\
A_{33} &= (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} &= -2
\end{aligned}
$$

Now evaluate the determinant by expanding along the first row (for example):

$$
\begin{aligned}
|A| &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\
&= 2 \times (-11) + 1 \times 3 + 1 \times 1 = -18.
\end{aligned}
$$

Then the inverse is given by

$$
\begin{aligned}
A^{-1} &= \frac{1}{|A|} adj A \\
&= \frac{1}{|A|}\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T \\
&= -\frac{1}{18}\begin{bmatrix} -11 & 3 & 1 \\ 5 & -3 & -7 \\ 4 & -6 & -2 \end{bmatrix}^T \\
&= -\frac{1}{18}\begin{bmatrix} -11 & 5 & 4 \\ 3 & -3 & -6 \\ 1 & -7 & -2 \end{bmatrix}
\end{aligned}
$$

2.3.4 The inverse of a product of matrices

If A and B are both non-singular $n \times n$ matrices, then

$$(AB)^{-1} = B^{-1}A^{-1}$$

To demonstrate the truth of this statement, we note that since $|A| \neq 0$ and $|B| \neq 0$ then $|AB| = |A||B| \neq 0$. Therefore AB is non-singular and has an inverse.

Also $(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$
Similarly $(B^{-1}A^{-1})(AB) = I$.

Hence $B^{-1}A^{-1} = (AB)^{-1}$.

More generally, if $A, B, \ldots, X, Y$ are all non-singular $n \times n$ matrices, then

$$(AB \ldots XY)^{-1} = Y^{-1}X^{-1} \ldots B^{-1}A^{-1}.$$

Note that the order of the terms is reversed when taking the inverse of products of matrices. different order of terms in the above.

Suppose A, B, C are all $n \times n$ matrices and A is non-singular. Then we have the results

1. $AB = 0 \quad \Rightarrow \quad B = 0$

   Proof: $AB = 0 \quad \Rightarrow \quad A^{-1}(AB) = (A^{-1}A)B = IB = 0 \quad \Rightarrow \quad B = 0.$

2. $AB = AC \quad \Rightarrow \quad B = C$

   Proof: $A^{-1}(AB) = A^{-1}(AC) \quad \Rightarrow \quad (A^{-1}A)B = (A^{-1}A)C \quad \Rightarrow \quad B = C.$

2.4 Systems of Linear Equations

Consider the system of n linear equations in n unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}$$

where the coefficients a_{ij} and b_k are all known constants.

As discussed before we have defined matrix multiplication so that we can write this system in matrix form

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

If **all** the elements of B are zero, the system of equations is called **homogeneous**.

If at least **one** of the elements of B is not zero, the system is called **non-homogeneous**.

2.4.1 Solution of a non-homogeneous system

If A is **non-singular**, then A^{-1} exists, and pre-multiplying the equation $AX = B$ by A^{-1} gives for the LHS

$$A^{-1}(AX) = (A^{-1}A)X = IX = X,$$

and hence

$$X = A^{-1}B.$$

We therefore have a **unique solution** of the system of equations.

Example: Find the solution of the following equations, using matrix methods.

$$\begin{aligned} 2x - 3y &= 3 \\ x + 4y &= 7 \end{aligned}$$

Solution: Here $A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}.$

$$|A| = 8 + 3 = 11$$

and

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T = \frac{1}{11} \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$$

The equations $AX = B$ have solution $X = A^{-1}B$.

$$\text{i.e.} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 12 + 21 \\ -3 + 14 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

i.e $x = 3$, $y = 1$ is the unique solution.

Exercise: Use matrix methods to find the solution of the following equations:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= -2 \\ -x_2 + 3x_3 &= -8 \\ x_1 + 4x_2 - x_3 &= 9. \end{aligned}$$

Solution: The equations can be written in the form $AX = B$ where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & 3 \\ 1 & 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 \\ -8 \\ 9 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The matrix A^{-1} was evaluated earlier and we found

$$A^{-1} = -\frac{1}{18}\begin{bmatrix} -11 & 5 & 4 \\ 3 & -3 & -6 \\ 1 & -7 & -2 \end{bmatrix}$$

Now, $AX = B \Rightarrow X = A^{-1}B$, and therefore

$$\begin{aligned} X &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = -\frac{1}{18}\begin{bmatrix} -11 & 5 & 4 \\ 3 & -3 & -6 \\ 1 & -7 & -2 \end{bmatrix}\begin{bmatrix} -2 \\ -8 \\ 9 \end{bmatrix} \\ &= -\frac{1}{18}\begin{bmatrix} 22-40+36 \\ -6+24-54 \\ -2+56-18 \end{bmatrix} \\ &= -\frac{1}{18}\begin{bmatrix} 18 \\ -36 \\ 36 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \end{aligned}$$

giving the unique solution $x_1 = -1$, $x_2 = 2$, $x_3 = -2$.

If A is **singular** and the system is non-homogeneous, then $|A| = 0$ and B has at least one non-zero element. Since $|A| = 0$, A^{-1} does not exist and there are two possibilities:

1. The system of equations is **inconsistent** and has **no solution**.
2. There are **infinitely** many solutions to the problem.

Example: Find values of α for which the following equations have solutions and find these solutions.

$$2x + y = 1 \tag{2.2}$$
$$4x + 2y = \alpha \tag{2.3}$$

Solution: Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. Then $|A| = 0$, so the system is singular.

It should be obvious by inspection that the given system has **no solutions** if $\alpha \neq 2$ (subtract twice the 1st equation from the 2nd equation).

If $\alpha = 2$ we have infinitely many solutions, since then both equations give exactly the same information.
To prove these results and find the solutions, we first specify one of the unknowns.
Let $x = \lambda$, where λ is an arbitrary constant. Then from (2.2)

$$y = 1 - 2x \quad = \quad 1 - 2\lambda \tag{2.4}$$

From (2.3):

$$2y = \alpha - 4x = \alpha - 4\lambda \quad \Rightarrow \quad y = \frac{\alpha}{2} - 2\lambda. \tag{2.5}$$

Now (2.4) and (2.5) must give the same value for y and, therefore,

$$\frac{\alpha}{2} - 2\lambda = 1 - 2\lambda \quad \Rightarrow \quad \alpha = 2.$$

Hence if $\alpha \neq 2$ there is **no solution** and the equations are **inconsistent**. If $\alpha = 2$ then $x = \lambda$, $y = 1 - 2\lambda$ is a solution for **all** values of λ and there are therefore an **infinite number of solutions**.

We can understand this example geometrically. The two equations (2.2) and (2.2) represent straight lines in the (x, y)-plane. Both have slope -2. If the lines do not intersect the y-axis at the same point, then they never intersect each other. This corresponds to the equations being inconsistent, and the case $\alpha \neq 2$. If $\alpha = 2$, the lines are coincident, and there are an infinite number of solutions to the equations.

Exercise: Consider the following system of equations

$$x + y + z \quad = \quad 3 \tag{2.6}$$
$$x - y + 3z \quad = \quad 5 \tag{2.7}$$
$$3x - y + 7z \quad = \quad \alpha \tag{2.8}$$

Find values of α for which these have solutions and find these solutions.

Solution: We might just spot that the lhs of row 3 = lhs of (row 1 + 2 $\times$ row 2) and conclude that we must have $\alpha = 3 + 2(5) = 13$. This gives no solutions if $\alpha \neq 13$ and an infinite number of solutions if $\alpha = 13$.
However if we fail to spot this relationship, we proceed using matrix methods as follows:
Let

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & -1 & 7 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -1 & 7 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} \\ &= -7 + 3 - (7 - 9) - 1 + 3 = -4 + 2 + 2 = 0. \end{aligned}$$

Since $|A| = 0$ we have either no solutions or an infinite number of solutions.

Let $x = \lambda$. Then from (2.21)

$$y + z = 3 - \lambda$$

and from (2.22)

$$-y + 3z = 5 - \lambda$$

and by addition this gives

$$4z = 8 - 2\lambda \quad \Rightarrow \quad z = 2 - \frac{\lambda}{2}.$$

From (2.21)

$$y = 3 - \lambda - z = 3 - \lambda - (2 - \frac{\lambda}{2}) = 1 - \frac{\lambda}{2}.$$

Therefore $x = \lambda$, $y = 1 - \frac{\lambda}{2}$, $z = 2 - \frac{\lambda}{2}$.

Substitute into (2.23) to get

$$3x - y + 7z = \alpha$$

and hence

$$3\lambda - (1 - \frac{\lambda}{2}) + 7(2 - \frac{\lambda}{2}) = \alpha \quad \Rightarrow \quad \alpha = 13.$$

Therefore if $\alpha = 13$ we have an infinite number of solutions $x = \lambda$, $y = 1 - \frac{\lambda}{2}$, $z = 2 - \frac{\lambda}{2}$ (λ arbitrary), whereas if $\alpha \neq 13$ there are no solutions.

2.4.2 Solution of a homogeneous systems of equations

In this case the equations are

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0.
\end{aligned}$$

We can write this system of equations in matrix notation as $AX = 0$,

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n3} \end{bmatrix}$, $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, and the 0 on the rhs is the $n \times 1$ zero matrix $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Clearly the system has the **trivial solution** $X = 0 \quad \Rightarrow \quad x_1 = x_2 = \cdots = x_n = 0.$

We now ask under what conditions are there any solutions apart from the trivial one.
If A is nonsigular, then $|A| \neq 0$, A^{-1} exists and the only solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus if $|A| \neq 0$, the trivial solution is the only solution.

If A is singular ($|A| = 0$), then there are infinitely many solutions.

Note that inconsistency is not a possibility in this case as we know we have at least one solution $X = 0$.

Example: Solve

$$\begin{aligned} 3x + 4y &= 0 \\ x - y &= 0 \end{aligned}$$

Solution: Let $A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix}$. Then $|A| = -3 - 4 = -7 \neq 0$,

so A is non-singular and the only solution is the trivial solution $x = y = 0$.

Exercise: Solve

$$\begin{aligned} x + 2y &= 0 \\ 2x + 4y &= 0 \end{aligned}$$

Solution: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

Then $|A| = 4 - 4 = 0$, and so A is singular and the system has infinitely many solutions.

Let $x = \lambda$, where λ is arbitrary. Then $2y = -x = -\lambda$, i.e. $y = -\dfrac{\lambda}{2}$.

Therefore $x = \lambda$, $y = -\dfrac{\lambda}{2}$ is a solution for all values of λ.

Exercise: Find the values of α such that the system

$$\begin{aligned} 4x + 3y &= 0 \\ 2x + \alpha y &= 0 \end{aligned}$$

has infinitely many solutions. Find these solutions.

Solution: Let $A = \begin{bmatrix} 4 & 3 \\ 2 & \alpha \end{bmatrix}$. Then $|A| = 4\alpha - 6$.

For the system to have infinitely many solutions, we require $|A| = 0$. Therefore $\alpha = 3/2$.

Now let $x = \lambda$. Hence $4x + 3y = 4\lambda + 3y = 0 \quad \Rightarrow \quad y = -\dfrac{4\lambda}{3}$.
Hence the required solutions are $x = \lambda$, $y = -\dfrac{4\lambda}{3}$, for all values of λ.

Example: Prove that the equations

$$\begin{aligned} x + y + z &= 0 && (2.9) \\ x - y + 3z &= 0 && (2.10) \\ 3x - y + 7z &= 0 && (2.11) \end{aligned}$$

have infinitely many solutions. Construct these solutions.

Solution: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 3 & -1 & 7 \end{bmatrix}$. Then $|A| = -7 + 3 - (7 - 9) - 1 + 3 = -4 + 2 + 2 = 0$.

Hence A is singular, and the system has infinitely many solutions. We proceed as before:
Let $x = \lambda$. Add (2.9) and (2.10) to obtain

$$2x + 4z = 0 \quad \Rightarrow \quad z = -\frac{x}{2} = -\frac{\lambda}{2}.$$

Then from (2.10) we obtain

$$y = x + 3z = \lambda - \frac{3\lambda}{2} = -\frac{\lambda}{2}.$$

Substituting in (2.11) we can confirm that

$$3x - y + 7z = 3\lambda + \frac{\lambda}{2} - \frac{7\lambda}{2} = 0,$$

and hence $x = \lambda$, $y = -\dfrac{\lambda}{2}$, $z = -\dfrac{\lambda}{2}$ satisfies all three equations for any value of λ.

2.4.3 Summary

A system of n linear equations in n unknowns can be expressed in matrix form as

$$AX = B.$$

If $B \neq 0$ the system is called **non-homogeneous**. If

1. $|A| \neq 0$, the system has the unique solution $X = A^{-1}B$;
2. $|A| = 0$, the system has either infinitely many solutions or no solutions at all.

If $B = 0$ the system is called **homogeneous**. If

1. $|A| \neq 0$, the system has only the trivial solution $X = 0$;
2. $|A| = 0$, the system has infinitely many (non-trivial) solutions.

2.5 Gaussian Elimination

Gaussian elimination is a convenient method that can be used to

1. solve a system of linear equations $AX = B$,
2. find the inverse of a matrix.

2.5.1 Solving systems of linear equations

Consider the system of equations:

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &= 3 \qquad (2.12) \\ 4x_1 + 7x_2 + 7x_3 &= 1 \qquad (2.13) \\ -2x_1 + 4x_2 + 5x_3 &= -7 \qquad (2.14) \end{aligned}$$

The following operations do not change the solution of the equations:

1. equations may be interchanged,
2. any equation may be multiplied by a non-zero constant,
3. any equation may be added to another equation.

In matrix form, these operations correspond to the following:

1. rows may be interchanged,
2. any row may be multiplied by a non-zero constant,

3. any row may be added to another row.

Gaussian elimination is a method for using these properties to solve the system by successively eliminating variables from the equations.

Step 1: Eliminate x_1 from (2.13) and (2.13):

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &= 3 && (2.12) \\ 3x_2 + x_3 &= -5 && (2.13) - 2 \times (2.12) && (2.15) \\ 6x_2 + 8x_3 &= -4 && (2.14) + (2.12) && (2.16) \end{aligned}$$

Step 2: Now eliminate x_2 from (2.16)

$$\begin{aligned} 2x_1 + 2x_2 + 3x_3 &= 3 && (2.12) \\ 3x_2 + x_3 &= -5 && (2.15) \\ 6x_3 &= 6 && (2.16) - 2 \times (2.15) && (2.17) \end{aligned}$$

The original system has now been reduced to a **triangular** system of equations.This can be solved by **back substitution**:

$$\begin{aligned} &(2.17) \quad \Rightarrow \quad \underline{x_3 = 1} \\ &(2.15) \quad \Rightarrow \quad 3x_2 + 1 = -5 \quad \Rightarrow \quad \underline{x_2 = -2} \\ &(2.12) \quad \Rightarrow \quad 2x_1 - 4 + 3 = 3 \quad \Rightarrow \quad \underline{x_1 = 2}. \end{aligned}$$

More generally, the system $AX = B$ can be reduced to a triangular set of equations which are easily solved by back substitution. In hand calculations this can be done in **tabular form** incorporating some simple but effective checks on the accuracy as in the following example.

Example: Use Gaussian Elimination to solve

$$\begin{aligned} x_1 + x_2 + 2x_3 - x_4 &= 5 \\ x_1 + 3x_2 + 2x_3 + x_4 &= 17 \\ 3x_1 + x_2 + 3x_3 + x_4 &= 18 \\ x_1 + 3x_2 + 4x_3 + 2x_4 &= 27 \end{aligned}$$

Solution: To solve in tabular form, we set up a table in the following way:

Row	x_1	x_2	x_3	x_4	B	Sum Check	Operation
(1)	1	1	2	-1	5		
(2)	1	3	2	1	17		
(3)	3	1	3	1	18		
(4)	1	3	4	2	27		

The rows correspond to the original system. The "Sum Check" column gives the sum of the numbers to its left, and helps to prevent arithmetic errors. Adding this in, we get

Row	x_1	x_2	x_3	x_4	B	Sum Check	Operation
* (1)	1	1	2	-1	5	8	
(2)	1	3	2	1	17	24	
(3)	3	1	3	1	18	26	
(4)	1	3	4	2	27	37	

The rows in subsequent tables correspond to the equations which are formed in the elimination process. The "Operation" column describes how the current row is formed from previous rows. The operation should also be applied to the sum check column and the result checked against the sum of the numbers to its left. This is equivalent to the algebraic operations we did before.

In each block of rows one row is indicated by * — this is called the **pivotal row**. In this example, the first row in each block has been chosen to be the pivotal row, but this is not necessary.

At each stage, multiples of the pivotal row are subtracted from/added to the other rows to eliminate x_1, then x_2 etc.

Row	x_1	x_2	x_3	x_4	B	Sum Check	Operation
* (5)		2	0	2	12	16	(2) – (1)
(6)		-2	-3	4	3	2	(3) – 3 $\times$ (1)
(7)		2	2	3	22	29	(4) – (1)
* (8)		0	-3	6	15	18	(6) + (5)
(9)		0	2	1	10	13	(7) – (5)
* (10)			0	5	20	25	(9) + $\frac{2}{3}\times$ (8)

The pivotal rows can be used to get the solution using back substitution:

$$\begin{array}{rcllcl}
(10) & \Rightarrow & 5x_4 = 20 & & \Rightarrow & x_4 = 4 \\
(8) & \Rightarrow & -3x_3 + 6x_4 = 15 & & \Rightarrow & x_3 = 3 \\
(5) & \Rightarrow & 2x_2 + 2x_4 = 12 & & \Rightarrow & x_2 = 2 \\
(1) & \Rightarrow & x_1 + x_2 + 2x_3 - x_4 = 5 & & \Rightarrow & x_1 = 1
\end{array}$$

2.5.2 Finding the inverse of a matrix

Suppose A is a non-singular $n \times n$ matrix, so A^{-1} exists. Let I be the $n \times n$ identity matrix and let B be an unknown $n \times n$ matrix, which satisfies

$$AB = I.$$

Then $B = A^{-1}I = A^{-1}$.
We can therefore find A^{-1} by using Gaussian elimination in tabular form to solve the system $AB = I$.

Basically, if we perform row operations to transform A into I, then if we perform the same operations on I, we will obtain A^{-1}. This is most easily seen by example.

Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 4 \\ 3 & 2 & 8 \end{bmatrix}$, and hence solve the system

$$\begin{aligned} x + 2y + z &= 1 \\ 2x + 2y + 4z &= 8 \\ 3x + 2y + 8z &= 17 \end{aligned}$$

Solution:

Row	A			I			Sum Check	Operation
∗ (1)	1	2	1	1	0	0	5	
(2)	2	2	4	0	1	0	9	
(3)	3	2	8	0	0	1	14	
(4)	1	2	1	1	0	0	5	(1)
∗ (5)	0	-2	2	-2	1	0	-1	(2) − 2 × (1)
(6)	0	-4	5	-3	0	1	-1	(3) − 3 × (1)
(7)	1	0	3	-1	1	0	4	(4) + (5)
(8)	0	-2	2	-2	1	0	-1	(5)
∗ (9)	0	0	1	1	-2	1	1	(6) − 2 ×(5)
(10)	1	0	0	-4	7	-3	1	(7) − 3 ×(9)
(11)	0	-2	0	-4	5	-2	-3	(8) − 2 ×(9)
(12)	0	0	1	1	-2	1	1	(9)
(13)	1	0	0	-4	7	-3	1	(10)
(14)	0	1	0	2	$-5/2$	1	$3/2$	(−1/2) × (11)
(15)	0	0	1	1	-2	1	1	(12)
					$\Uparrow$ $=A^{-1}$			

You should check by multiplication that this does indeed yield the inverse of A.
Now we solve the equations:

$$\begin{aligned}
X &= A^{-1}B \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -4 & 7 & -3 \\ 2 & -5/2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 8 \\ 17 \end{bmatrix} \\
&= \begin{bmatrix} -4 \times 1 + 7 \times 8 - 3 \times 17 \\ 2 \times 1 - (5/2) \times 8 + 1 \times 17 \\ 1 \times 1 - 2 \times 8 + 1 \times 17 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.
\end{aligned}$$

2.6 Eigenvalues and Eigenvectors

Suppose A is a square matrix and X is a column matrix (i.e. vector) with *at least one non-zero entry* such that

$$AX = \lambda X, \tag{2.18}$$

where λ is a scalar.

The scalar λ is called an **eigenvalue** or (much less commonly) eigenroot or latent root. The vector X is called an **eigenvector** corresponding to the eigenvalue λ.

Clearly kX is also an eigenvector for any non-zero scalar k.

(2.18) can be written as:

$$\begin{aligned}
AX &= \lambda I X \\
\text{i.e.} \quad (A - \lambda I)X &= 0
\end{aligned} \tag{2.19}$$

.
Now, $(A - \lambda I)X = 0$ has a non-trivial solution for the column vector X **if and only if**

$$|A - \lambda I| = 0. \tag{2.20}$$

Therefore, to find the eigenvalues of A, we need to solve (2.20) for λ.
Equation (2.20) is called the **characteristic equation** for the matrix A.

For any eigenvalue λ (i.e. a value of λ satisfying (2.20)), we can find a corresponding column vector X, called the eigenvector, satisfying (2.18).

If A is an $n \times n$ matrix, so that $|A - \lambda I|$ is an n^{th} order determinant, then $|A - \lambda I|$ is a polynomial of degree n in λ. We therefore solve the characteristic equation by finding the roots of this polynomial. There will be therefore be at most n distinct solutions λ to (2.20).

Example: Find the eigenvalues and corresponding eigenvectors associated with the matrix

$$A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}.$$

Solution: $A = \begin{bmatrix} 4 & 3 \\ -2 & -1 \end{bmatrix}$ and $\lambda I = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$.

Therefore

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 3 \\ -2 & -1-\lambda \end{vmatrix} \\ &= (4-\lambda)(-1-\lambda) - 3 \times (-2) \\ &= -4 + \lambda - 4\lambda + \lambda^2 + 6 \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 2)(\lambda - 1). \end{aligned}$$

Therefore $|A - \lambda I| = 0 \quad \Rightarrow \quad \lambda = 2$ or $\lambda = 1$.

Therefore the eigenvalues of A are $\lambda = 2$ and $\lambda = 1$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigenvector. We need to find an X for each of the eigenvalues. We take each eigenvalue in turn:

Case $\lambda = 1$: In this case $(A - \lambda I)X = 0$ gives

$$(A - I)X = \begin{bmatrix} 3 & 3 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $3x_1 + 3x_2 = 0 \quad \Rightarrow \quad x_2 = -x_1$.
Note that we could also have used $-2x_1 - 2x_2 = 0$; the answer is the same.
Putting $x_1 = \alpha$, an eigenvector corresponding to $\lambda = 1$ for all $\alpha \neq 0$ is:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Case $\lambda = 2$: In this case $(A - \lambda I)X = 0$ gives

$$(A - I)X = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

i.e. $2x_1 + 3x_2 = 0 \quad \Rightarrow \quad x_2 = -\dfrac{2}{3}x_1.$

Putting $x_1 = 3\beta$, an eigenvector corresponding to $\lambda = 2$ for all $\beta \neq 0$ is:

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3\beta \\ -2\beta \end{bmatrix} = \beta \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Sometimes it is required that we **normalise** the eigenvectors — i.e. that we find eigenvectors of unit length:

$$X_N = \frac{X}{|X|}.$$

For case $\lambda = 1$, the normalised eigenvector is

$$X_N = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

since $|X| = \sqrt{\alpha^2 + \alpha^2} = \sqrt{2}\alpha$

For case $\lambda = 2$, the normalised eigenvector is

$$X_N = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ -2 \end{bmatrix},$$

since $|X| = \beta\sqrt{3^2 + (-2)^2} = \beta\sqrt{13}.$

Exercise: Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 2 & 2 & 3 \\ -2 & 3 & 2 \end{bmatrix}.$$

Solution:
First find the eigenvalues by solving the characteristic equation:

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 & -2 \\ 2 & 2-\lambda & 3 \\ -2 & 3 & 2-\lambda \end{vmatrix}.$$

Expand the determinant by row 1:

$$
\begin{aligned}
|A-\lambda I| &= (1-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{vmatrix} - (-2)\begin{vmatrix} 2 & 3 \\ -2 & 2-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 2-\lambda \\ -2 & 3 \end{vmatrix} \\
&= (1-\lambda)[(2-\lambda)^2-9] + 2[2(2-\lambda)+6] - 2[6+2(2-\lambda)] \\
&= (1-\lambda)[4-4\lambda+\lambda^2-9] + 2(10-2\lambda) - 2(10-2\lambda) \\
&= (1-\lambda)(\lambda^2-4\lambda-5) \\
&= (1-\lambda)(\lambda-5)(\lambda+1).
\end{aligned}
$$

Thus $|A-\lambda| = 0 \Rightarrow (1-\lambda)(\lambda-5)(\lambda+1) = 0$, and the eigenvalues are $\lambda = -1, 1, 5$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigenvector, and consider the three distinct eigenvalues:

<u>*Case* $\lambda = -1$:</u> In this case

$$
(A-\lambda I)X = \begin{bmatrix} 2 & -2 & -2 \\ 2 & 3 & 3 \\ -2 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
$$

and so

$$
\begin{aligned}
2x_1 - 2x_2 - 2x_3 &= 0 \qquad (2.21) \\
2x_1 + 3x_2 + 3x_3 &= 0 \qquad (2.22) \\
-2x_1 + 3x_2 + 3x_3 &= 0 \qquad (2.23)
\end{aligned}
$$

Subtracting (2.23) from (2.22) gives $4x_1 = 0 \Rightarrow x_1 = 0$.

Substituting in any of (2.21)–(2.23) then gives $x_2 + x_3 = 0$.

Putting $x_2 = \alpha$ for any $\alpha \neq 0 \Rightarrow x_3 = -\alpha$.

Hence $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha \\ -\alpha \end{bmatrix} = \alpha \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = -1$.

<u>*Case* $\lambda = 1$:</u> In this case

$$
(A-\lambda I)X = \begin{bmatrix} 0 & -2 & -2 \\ 2 & 1 & 3 \\ -2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
$$

and so

$$\begin{aligned} -2x_2 - 2x_3 &= 0 \quad (2.24)\\ 2x_1 + x_2 + 3x_3 &= 0 \quad (2.25)\\ -2x_1 + 3x_2 + x_3 &= 0 \quad (2.26) \end{aligned}$$

From (2.24), $x_2 + x_3 = 0 \;\Rightarrow\; x_3 = -x_2$.

Putting $x_1 = \beta$ for any $\beta \neq 0$, (2.25) $\Rightarrow$

$$2\beta + x_2 + 3(-x_2) = 0 \;\Rightarrow\; x_2 = \beta.$$

Hence $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \beta \\ \beta \\ -\beta \end{bmatrix} = \beta \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 1$.

<u>*Case* $\lambda = 5$:</u> In this case

$$(A - \lambda I)X = \begin{bmatrix} -4 & -2 & -2 \\ 2 & -3 & 3 \\ -2 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and so

$$\begin{aligned} -4x_1 - 2x_2 - 2x_3 &= 0 \quad (2.27)\\ 2x_1 - 3x_2 + 3x_3 &= 0 \quad (2.28)\\ -2x_1 + 3x_2 - 3x_3 &= 0 \quad (2.29) \end{aligned}$$

$$\begin{aligned} \text{i.e.} \quad 2x_1 + x_2 + x_3 &= 0 \qquad -(1/2) \times (2.27) \quad (2.30)\\ \text{and} \quad 2x_1 - 3x_2 + 3x_3 &= 0 \qquad -1 \times (2.29) \quad (2.31) \end{aligned}$$

Let $x_1 = \gamma \neq 0$. Then (2.30) $\Rightarrow$ $2\gamma + x_2 + x_3 = 0$.

Adding $3\times$ this equation to (2.31) $\Rightarrow$ $8\gamma + 6x_3 = 0 \;\Rightarrow\; x_3 = -\dfrac{4\gamma}{3}$.

(2.30) then $\Rightarrow$ $x_2 = -\dfrac{2\gamma}{3}$.

Setting $3\gamma_0 = \gamma$, we see that

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \gamma_0 \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda = 5$.

3 Ordinary Differential Equations

A differential equation is an equation which involves derivatives of an unknown function. For example

$$\frac{dy}{dx} = x^5 + 5, \tag{3.1}$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = x^2, \tag{3.2}$$

$$\frac{d^3y}{dx^3} = x + y\frac{dy}{dx}. \tag{3.3}$$

In these examples, y is an unknown function of x (the **independent variable**). When there is only one independent variable, the equation is described as an **ordinary differential equation** (conventionally abbreviated to **ODE**). Our objective in this Section will be to study methods for finding solutions $y(x)$ to a few classes of ODE.

We know that we can also have functions of more than one variable. Such functions have partial derivatives, and we often have equations involving these partial derivatives. For example, if z is a function of two independent variables x and y then z could satisfy the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$$

This is an example of a **partial differential equation**. Such equations are central to understanding the behaviour of systems whose behaviour depends on both space and time (for example) but will not be considered further in this module.

The **order** of a differential equation is determined by the order of the highest derivative. Thus (3.1) is first order $\left(\frac{dy}{dx}\right)$, (3.2) is second order $\left(\frac{d^2y}{dx^2}\right)$ and (3.3) is third order $\left(\frac{d^3y}{dx^3}\right)$. We will be studying first and second order equations in this module.

(3.1) and (3.2) are classed as **linear**, since they do not contain any products of y and/or its derivatives. However, (3.3) is **non-linear** because of the term $y\frac{dy}{dx}$, which is a product of y and its first derivative.

Differential equations arise naturally when modelling physical phenomena. For example, consider a particle of mass m being accelerated by a given force F. We can calculate its acceleration, a, using Newton's 2nd Law of Motion which can be written as the **algebraic** equation: $F = ma$. Noting that acceleration is the rate of change of speed (v), we can instead write this as a 1st order ODE:

$$F = m\frac{dv}{dt},$$

or as a 2nd order ODE (using $v = \dfrac{dx}{dt}$, where x is the position of the particle):

$$F = m\frac{d^2x}{dt^2}.$$

3.1 First Order Linear Equations

The general form of a first order linear ODE is

$$\frac{dy}{dx} = f\left(x, y\right).$$

The majority of ODEs of this form do **not** usually have an **analytic** solution (i.e. it is not possible to give an explicit formula for the answer $y(x)$). However, there are many important cases where a solution can be found, and we will now focus on these.

3.1.1 Direct Integration

Consider first equations of the form

$$\boxed{\frac{dy}{dx} = f(x)}$$

This type of equation can be solved immediately by integration. The general solution is

$$y = \int f(x)\, dx + C,$$

where C is an arbitrary constant.

Example: Solve $\dfrac{dy}{dx} = 2 + 3x^2$ given that $y(0) = 2$.

Solution: The general solution is

$$y(x) = \int \left(2 + 3x^2\right) dx + C = 2x + x^3 + C.$$

Now, $y(0) = C$ and so if $y(0) = 2$, then we must have $C = 2$, and so the required solution is

$$y(x) = 2x + x^3 + 2.$$

Note that without specifying the condition $y(0) = 2$, the general solution of the ODE is actually a **family** of functions $y(x)$, distinguished by the value of C. The condition $y(0) = 2$ is an example of an **initial condition**. The combination of an ODE and an initial condition is referred to as an **initial value problem**.

3.1.2 Separation of Variables

Now consider equations of the form

$$\boxed{\frac{dy}{dx} = f(x)g(y)}$$

It is possible to re-write equations of this form so that only x occurs on one side of the equation and only y on the other. In this new form both sides can be integrated to give the solution.

NOTE. Having performed the integrations, one should always manipulate the solution into the form $y = F(x)$ if it is possible to do so.

If $\dfrac{dy}{dx} = f(x)g(y)$ then $\dfrac{1}{g(y)}\dfrac{dy}{dx} = f(x)$ and so

$$\int \frac{dy}{g(y)} = \int f(x)\,dx.$$

Example: Solve $\dfrac{dy}{dx} = xe^{-y}$.

Solution: The equation can be re-written to give

$$e^y\frac{dy}{dx} = x \quad \Rightarrow \quad \int e^y dy = \int x dx \quad \Rightarrow \quad e^y = \frac{x^2}{2} + C.$$

Taking logs of both sides we have:

$$\ln e^y = \ln\left(\frac{x^2}{2} + C\right) \quad \Rightarrow \quad y = \ln\left(\frac{x^2}{2} + C\right).$$

Example: Solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{4y}{x(y-3)} \qquad (y > 0).$$

Solution: Re-arrange the equation to give

$$\left(\frac{y-3}{y}\right)\frac{dy}{dx} = \frac{4}{x} \quad \Rightarrow \quad \int\left(1 - \frac{3}{y}\right)dy = \int \frac{4}{x}dx$$

and so the general solution is

$$y - 3\ln y = 4\ln x + C.$$

Note that strictly speaking there should be an integration constant for both integrals. However, these can be combined into a single constant C (one constant – another constant = yet another constant).

With a little work, we can cast the general solution in a neater form. On the right hand side we have $4\ln x + C = \ln x^4 + C = \ln x^4 + \ln A = \ln(Ax^4)$, where $C = \ln A$, since any constant can be written as the log of another constant.
On the left hand side we have $y - 3\ln y = y + \ln y^{-3} = \ln(e^y) + \ln y^{-3} = \ln(e^y y^{-3})$.
Hence, taking exponentials of both sides, we have

$$e^y y^{-3} = Ax^4 \quad \Rightarrow \quad e^y = Ax^4 y^3,$$

where A is an arbitrary constant. Note that in this case it is impossible to write the answer in the form "y is a function of x".

3.1.3 Integrating Factors

Now consider a general first order linear ODE of the form

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)} \tag{3.4}$$

where $P(x)$ and $Q(x)$ are functions of x only.
The method that we use to solve such equations is to multiply the equation by a new function, the **integrating factor**, that makes the l.h.s. the derivative of a product.

The integrating factor (IF) for the equation $\dfrac{dy}{dx} + P(x)y = Q(x)$ is

$$I(x) = \exp\left(\int P(x)\,dx\right).$$

If we multiply Eq.(3.4) by $I(x)$, it becomes

$$\frac{d\left(y(x)I(x)\right)}{dx} = Q(x)I(x),$$

which can be integrated directly to give the solution

$$y(x)I(x) = \int Q(x)I(x)\ dx \quad \Rightarrow \quad y(x) = \frac{1}{I(x)}\int Q(x)I(x)\ dx.$$

To verify that $\dfrac{d\left(y(x)I(x)\right)}{dx} = Q(x)I(x)$ is equivalent to the original equation, we note first that

$$\frac{dI}{dx} = \frac{d}{dx}\left(\exp\left(\int P(x)\,dx\right)\right) = P(x)\exp\left(\int P(x)\,dx\right) = P(x)I(x).$$

Then

$$\frac{d\left(y(x)I(x)\right)}{dx} = I(x)\frac{dy}{dx} + y(x)\frac{dI}{dx} = I(x)\frac{dy}{dx} + y(x)P(x)I(x) = I(x)\left(\frac{dy}{dx} + P(x)y(x)\right),$$

and so $I(x)\left(\frac{dy}{dx}+P(x)y(x)\right)=Q(x)I(x)$ and dividing by $I(x)$ gives the original ODE.

Example: Solve the first order linear ODE $\quad \dfrac{dy}{dx}+2xy=4x.$

Solution: Using the notation introduced above, $P(x)=2x$ and $Q(x)=4x$. The integrating factor is therefore given by

$$I(x)=\exp\left(\int P(x)\ dx\right)=\exp\left(\int 2x\ dx\right)=\exp\left(x^2\right)=e^{x^2}.$$

Note that it is not necessary to include a constant of integration in the integrating factor because this would just multiply both sides of the ODE by a constant.

Multiplying the ODE by $I(x)$, we have

$$e^{x^2}\frac{dy}{dx}+2xe^{x^2}y=4xe^{x^2},$$

which can be written as

$$\frac{d}{dx}\left(e^{x^2}y\right)=4xe^{x^2}.$$

Now integrate both sides to get

$$e^{x^2}y=\int 4xe^{x^2}\ dx=2e^{x^2}+C,$$

where C is an arbitrary constant, or

$$y(x)=2+C\exp\left(-x^2\right).$$

Exercise: Use an integrating factor to find the general solution of the ODE

$$\frac{dy}{dx}-\frac{4}{x}y=4x^3.$$

Solution: $P(x)=-\dfrac{4}{x}$ [take care of the minus sign!] and so the integrating factor is

$$I(x)=\exp\left(\int -\frac{4}{x}dx\right)=\exp\left(-4\ln x\right)=\exp\left(\ln x^{-4}\right)=\frac{1}{x^4}.$$

Multiply the ODE by $I(x)$ to get

$$\frac{1}{x^4}\frac{dy}{dx}-\frac{4}{x^5}y=\frac{4}{x}\quad\Rightarrow\quad\frac{d}{dx}\left(\frac{y}{x^4}\right)=\frac{4}{x}.$$

Now integrate to obtain

$$\frac{y}{x^4}=\int\frac{4}{x}\ dx=4\ln x+C\quad\Rightarrow\quad y(x)=x^4\left(4\ln x+C\right).$$

Sometimes it is necessary to rewrite or transform the equation in to standard form before the integrating factor method can be applied.

Example: Solve the initial value problem $x\dfrac{dy}{dx} - y = x + 1, \quad y(1) = 2.$

Solution: The ODE is not in the standard form, and we must divide by x before attempting to find the integrating factor. Dividing by x, we have

$$\frac{dy}{dx} - \frac{1}{x}y = 1 + \frac{1}{x},$$

which is in standard form with $P(x) = -\dfrac{1}{x}$. Thus the integrating factor is

$$I(x) = \exp\left(\int -\frac{1}{x}dx\right) = \exp\left(-\ln x\right) = \exp\left(\ln x^{-1}\right) = \frac{1}{x}.$$

Multiplying the standard form ODE by $I(x)$, we obtain

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \frac{1}{x} + \frac{1}{x^2} \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x} + \frac{1}{x^2}.$$

Integrating gives the general solution to the ODE:

$$\frac{y}{x} = \int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx = \ln x - \frac{1}{x} + C \quad \Rightarrow \quad y(x) = x\ln x - 1 + Cx.$$

If $y(1) = 2$ then $2 = 1 \cdot 0 - 1 + C \quad \Rightarrow \quad C = 3$ and so the solution to the initial value problem is

$$y(x) = x\ln x - 1 + 3x.$$

3.2 Second Order Linear Equations with Constant Coefficients

We now turn our attention to second order linear ODEs. The most general form of linear ODE of nth order is:

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_1\frac{dy}{dx} + a_0y = f(x),$$

where the coefficients $a_i, \ i = 0, 1, \ldots, n$ do not depend on y, $a_n \neq 0$ and $f(x)$ is a function of x only. If the coefficients a_i do not depend on x (i.e. they are constants), then we say the equation has **constant coefficients**. We shall only deal explicitly with the case when $n = 2$ (second order equations), but the methods we develop also apply for $n > 2$.

Note that if an ODE of this form is a model for a physical system, then $f(x)$ represents the forcing of the system, while the left hand side of the ODE represents the system response.

3.2.1 Solution of the Homogeneous Equation

If $f(x) = 0$, the ODE is said to be **homogeneous**.
The general form of a second order homogeneous ODE is

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0,$$

where a_0, a_1 and a_2 are constants. To neaten things up, we will sometimes use the following alternative notation:

$$y' \equiv \frac{dy}{dx}, \quad y'' \equiv \frac{d^2y}{dx^2},$$

in which case, the general form is

$$a_2 y'' + a_1 y' + a_0 y = 0. \tag{3.5}$$

The method of solution that we will employ depends on an important feature of linear homogeneous equations:

> If y_1 and y_2 are two distinct solutions of (3.5), then so is $\alpha y_1 + \beta y_2$ for any constants α and β.

So, to proceed, we can seek *any* solutions of (3.5), and then combine them into a general solution. The types of solution that we seek are of the form

$$y = e^{kx}, \quad \text{where } k \text{ is a constant.}$$

Now

$$y = e^{kx} \quad \Rightarrow \quad y' = ke^{kx} = ky \quad \text{and} \quad y'' = k^2 e^{kx} = k^2 y.$$

and so (3.5) becomes

$$a_2 k^2 y + a_1 k y + a_0 y = (a_2 k^2 + a_1 k + a_0) y = 0.$$

and since $y \neq 0$ it follows that

$$\boxed{a_2 k^2 + a_1 k + a_0 = 0}$$

This is called the **auxiliary equation**.

The auxiliary equation is a quadratic equation, which always has two solutions (roots). Let the roots be k_1 and k_2. We need to consider three distinct cases:

1. k_1 and k_2 are **real and distinct** (i.e. $k_1 \neq k_2$).
2. k_1 and k_2 are **complex**.
3. k_1 and k_2 are **real and equal** (i.e. $k_1 = k_2$).

Case 1: k_1 and k_2 are real and distinct

In this case, we have shown that we have two distinct solutions to (3.5):

$$y_1 = e^{k_1 x} \quad \text{and} \quad y_2 = e^{k_2 x}.$$

The general solution to (3.5) is therefore

$$\boxed{y = A_1 e^{k_1 x} + A_2 e^{k_2 x}}$$

where A_1 and A_2 are arbitrary constants.

Example: Find the general solution of $y'' + 5y' + 6y = 0$.

Solution: The auxiliary equation is $k^2 + 5k + 6 = 0$. This may be written down directly or, if forgotten, substitute $y = e^{kx}$ into the given equation.
Now

$$\begin{aligned} k^2 + 5k + 6 = 0 \quad &\Rightarrow \quad (k+3)(k+2) = 0 \\ &\Rightarrow \quad k = -2 \quad \text{or} \quad -3. \end{aligned}$$

The roots are real and distinct, so the general solution for y is

$$y = A_1 e^{-2x} + A_2 e^{-3x},$$

where A_1 and A_2 are arbitrary constants.

Exercise: Find the general solution of $y'' - a^2 y = 0$, where a is a non-zero constant.

Solution: The auxiliary equation is $k^2 - a^2 = 0$ and

$$\begin{aligned} k^2 - a^2 = 0 \quad &\Rightarrow \quad (k+a)(k-a) = 0 \\ &\Rightarrow \quad k = -a \quad \text{or} \quad a \end{aligned}$$

The roots are real and distinct, so the general solution for y is

$$y = A_1 e^{-ax} + A_2 e^{ax}.$$

We can write this solution in a different form using the definitions of $\cosh ax$ and $\sinh ax$, and the relations

$$\begin{aligned} e^{ax} &= \cosh ax + \sinh ax \\ e^{-ax} &= \cosh ax - \sinh ax \end{aligned}$$

Substituting these into the expression for y the solution can also be written in the form

$$y = B_1 \cosh ax + B_2 \sinh ax,$$

where B_1 and B_2 are arbitrary constants. Either form is equally good.

Case 2: k_1 and k_2 are complex

The solutions $y(x)$ that we find using the auxiliary equation are now complex, but we usually require **real** solutions $y(x)$ to (3.5). We therefore have to impose a restriction on the arbitrary constants in the general solution.

Note that when k_1 and k_2 are complex, they are always a **complex conjugate pair** $a \pm ib$.

Example: Find the general solution of $y'' + 4y' + 13y = 0$.

Solution: The auxiliary equation is $k^2 + 4k + 13 = 0$, which has complex roots $k = -2 \pm 3i$. Thus the general solution can be written as

$$\begin{aligned} y &= A_1 e^{(-2+3i)x} + A_2 e^{(-2-3i)x} \\ &= e^{-2x}\left(A_1 e^{3ix} + A_2 e^{-3ix}\right), \end{aligned}$$

where A_1 and A_2 are arbitrary **complex** constants.
Now $e^{i\theta} = \cos\theta + i\sin\theta$ and so

$$\begin{aligned} y &= e^{-2x}\left(A_1(\cos 3x + i\sin 3x) + A_2(\cos 3x - i\sin 3x)\right) \\ &= e^{-2x}\left((A_1 + A_2)\cos 3x + i\,(A_1 - A_2)\sin 3x\right) \\ &= e^{-2x}\left(B_1\cos 3x + B_2\sin 3x\right) \end{aligned}$$

where $B_1 = A_1 + A_2$ and $B_2 = i(A_1 - A_2)$.
Therefore, it is necessary to choose A_1 and A_2 to be a complex conjugate pair, since this guarantees that B_1 and B_2 are real, and consequently the general solution y is real.

When deriving the general solution, it is not necessary to go through the whole process in this example every time. If the roots of the auxiliary equation are

$$k = a \pm ib$$

then the general solution of the differential equation can be written directly as

$$\boxed{y = e^{ax}\left(A_1\cos bx + A_2\sin bx\right)}$$

where A_1 and A_2 are arbitrary (real) constants.

Case 3: k_1 and k_2 are equal

If both roots of the auxiliary equation are equal, then they must be real. In this case, the approach used in the previous two cases does not give us two *distinct* solutions y.
The appropriate general solution in this case (if $k_1 = k_2 = a$) is

$$\boxed{y = e^{ax}\left(A + Bx\right)} \qquad (3.6)$$

where A and B are arbitrary constants.

Example: Find the general solution of $y'' - 4y' + 4y = 0$.

Solution: The auxiliary equation is

$$k^2 - 4k + 4 = 0 \quad \Rightarrow \quad (k-2)^2 = 0 \quad \Rightarrow \quad k = 2.$$

The required general solution is therefore

$$y = e^{2x}(A + Bx),$$

where A and B are arbitrary constants.

To see why (3.6) is the appropriate general solution, consider the general homogeneous ODE

$$a_2 y'' + a_1 y' + a_0 y = 0$$

which has auxiliary equation

$$a_2 k^2 + a_1 k + a_0 = 0.$$

In order for this equation to have two equal roots $k_1 = k_2$, we must have

$$a_1^2 = 4a_2 a_0.$$

Taking out a factor of $a_2 \neq 0$, the ODE can be rewritten as

$$y'' + \alpha y' + \beta y = 0,$$

where $\alpha = a_1/a_2$ and $\beta = a_0/a_2$. The condition for equal roots is then $\alpha^2 = 4\beta$. Now, if we introduce the new variable $z(x)$ defined by $z(x) = y' + \dfrac{\alpha}{2}y$, then z satisfies the equation

$$z' + \frac{\alpha}{2}z = y'' + \alpha y + \frac{\alpha^2}{4}y.$$

If we have equal roots ($\alpha^2 = 4\beta$), then z satisfies the first order homogeneous ODE

$$z' + \frac{\alpha}{2}z = 0,$$

which has general solution $z(x) = Ae^{-\alpha x/2}$. Recalling the definition of z, y therefore satisfies the first order ODE

$$y' + \frac{\alpha}{2}y = Ae^{-\alpha x/2},$$

which we can solve using an integrating factor (see above) to find

$$y(x) = (Ax + B)e^{-\alpha x/2},$$

which is the general solution of the original second order ODE for y.

3.2.2 Solution of Non-Homogeneous Equations

We now turn our attention to non-homogeneous equations:

$$a_2 y'' + a_1 y' + a_0 y = f(x), \quad f(x) \neq 0.$$

We find the general solution of such equations in a two-stage process:

1. Solve the associated *homogeneous* equation with $f(x) = 0$, as in the previous section. The solution, y_c of this homogeneous equation is called the **complementary function**. It will contain two arbitrary constants, as above.

2. Find **any** solution of the original equation (with $f(x) \neq 0$). This solution, y_p, will contain no arbitrary constants and is called a **particular solution** (or sometimes **particular integral**). There is no single method for finding a particular solution — it typically depends on an 'educated guess' based on your knowledge and experience of derivatives and integrals.

Having found y_c and y_p, the *general solution* of the given non-homogeneous equation is

$$y = y_c + y_p.$$

It contains the two arbitrary constants that come from the complementary function.

Finding the Particular Solution y_p

Finding the particular solution for a general $f(x)$ is a non-trivial problem. However, there are some standard methods that apply to a few specific forms of $f(x)$.

1. If $f(x)$ is a polynomial, then choose y_p to be a polynomial of the same degree.

2. If $f(x)$ is an exponential, $f(x) = ae^{\kappa x}$, then

   (a) if κ is **not** a root of the auxiliary equation, choose $y_p = be^{\kappa x}$, and find b by substituting into the ODE.

   (b) If the auxiliary function has two distinct roots, and κ is one of these roots, choose $y_p = bxe^{\kappa x}$, and find b by substituting into the ODE.

   (c) If the auxiliary function has a single repeated root, and this is equal to κ, choose $y_p = bx^2 e^{\kappa x}$, and find b by substituting into the ODE.

3. If $f(x)$ is a polynomial multiplied by an exponential (e.g. $f(x) = e^{2x}(3x^2 + 6)$), then try a solution of the same form (i.e. $y_p = e^{2x}(ax^2 + bx + c)$). This will work unless one (or more) of the terms in y_p occurs in y_c. In this case, then the order of the polynomial must be increased.

4. If $f(x)$ is a trigonometric function ($f(x) = a\cos(\kappa x) + b\sin(\kappa x)$), then choose $y_p = \alpha\cos(\kappa x) + \beta\sin(\kappa x)$, and find α and β by substituting in the ODE.

Example: Find the general solution of the ODE $y'' + 3y' + 2y = 6x^3$.

Solution: The auxiliary equation for $y'' + 3y' + 2y = 0$ is

$$k^2 + 3k + 2 = 0 \quad \Rightarrow \quad (k+2)(k+1) = 0$$

which has roots $-1, -2$. Thus the complementary function is $y_c = Ae^{-x} + Be^{-2x}$, where A and B are arbitrary constants.
Since $f(x) = 6x^3$ is a polynomial of order 3, for a particular solution we try

$$\begin{aligned} & y_p &=& \; ax^3 + bx^2 + cx + d \\ \Rightarrow \quad & y_p' &=& \; 3ax^2 + 2bx + c \\ \Rightarrow \quad & y_p'' &=& \; 6ax + 2b \end{aligned}$$

Substituting these into $y'' + 3y' + 2y = 6x^3$ gives

$$\begin{aligned} (6ax + 2b) + 3(3ax^2 + 2bx + c) + 2(ax^3 + bx^2 + cx + d) &= 6x^3 \\ \Rightarrow \quad 2ax^3 + (9a + 2b)x^2 + (6a + 6b + 2c)x + (2b + 3c + 2d) &= 6x^3. \end{aligned}$$

The coefficients of each power of x on the lhs and rhs of this equation must be equal, so

$$\begin{aligned} x^3 &: \; 2a = 6, \\ x^2 &: \; 9a + 2b = 0, \\ x^1 &: \; 6a + 6b + 2c = 0, \\ x^0 &: \; 2b + 3c + 2d = 0. \end{aligned}$$

The solution of this set of equation is

$$a = 3, \quad b = -\frac{27}{2}, \quad c = \frac{63}{2}, \quad d = -\frac{135}{4}.$$

Therefore $y_p = 3x^3 - \frac{27}{2}x^2 + \frac{63}{2}x - \frac{135}{4}$, and the general solution of the ODE is

$$\begin{aligned} y(x) &= y_c + y_p \\ &= Ae^{-x} + Be^{-2x} + 3x^3 - \frac{27}{2}x^2 + \frac{63}{2}x - \frac{135}{4}, \end{aligned}$$

where A and B are arbitrary constants.

Example: Find the general solution of the ODE $y'' - y' - 6y = 2e^{4x}$.

Solution: The auxiliary equation is $k^2 - k - 6 = 0$, which has roots $k = 3$ and $k = -2$. Thus the complementary function is $y_c = Ae^{3x} + Be^{-2x}$, where A and B are arbitrary constants.
Since $f(x)$ is an exponential function that does not appear in y_c, we try $y_p = ae^{4x}$ as the particular solution. Then

$$y_p = ae^{4x} \quad \Rightarrow \quad y_p' = 4ae^{4x} \quad \Rightarrow \quad y_p'' = 16ae^{4x}.$$

Substituting into the ODE gives

$$(16-4-6)ae^{4x} = 2e^{4x} \quad \Rightarrow \quad a = \frac{1}{3}.$$

Thus $y_p = \dfrac{e^{4x}}{3}$ and the general solution to the ODE is

$$y(x) = y_p + y_c = \frac{e^{4x}}{3} + Ae^{3x} + Be^{-2x},$$

where A and B are arbitrary constants.

Example: Find the general solution of the ODE $y'' - 2y' + y = 3x^2e^x$.

Solution: The auxiliary equation is $k^2 - 2k + 1 = 0$ which has a repeated root $k = 1$. Thus the complementary solution is

$$y_c = e^x(A + Bx),$$

where A and B are arbitrary constants.
Because $f(x) = 3x^2e^x$ is a polynomial of degree 2 multiplied by e^x, we might first think of trying a polynomial of degree 2 multiplied by e^x for the particular integral — i.e. $y_p = e^x\left(a + bx + cx^2\right)$. However the first two terms of y_p are of the same form as the terms of y_c, so we need to increase the order of the polynomial to 4. We therefore try

$$\begin{aligned}
y_p &= e^x\left(ax^2 + bx^3 + cx^4\right) \\
\Rightarrow \quad y_p' &= e^x(ax^2 + bx^3 + cx^4) + e^x(2ax + 3bx^2 + 4cx^3) \\
\Rightarrow \quad y_p'' &= e^x(ax^2 + bx^3 + cx^4) + 2e^x(2ax + 3bx^2 + 4cx^3) + e^x(2a + 6bx + 12cx^2)
\end{aligned}$$

Substituting this into the ODE and equating the coefficients of x^ne^x gives

$$\begin{aligned}
x^4e^x &: \text{ coefficient} = 0, \\
x^3e^x &: \text{ coefficient} = 0, \\
x^2e^x &: 12c = 3 \quad \Rightarrow \quad c = \frac{1}{4}, \\
x^1e^x &: b = 0, \\
e^x &: a = 0.
\end{aligned}$$

The general solution of the ODE is therefore

$$y(x) = y_c + y_p = e^x\left(A + Bx + \frac{x^4}{4}\right),$$

where A and B are arbitrary constants.

Example: Find the general solution of the ODE $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 13y = 20\cos 3t$.

Note that we are using t as the independent variable in this example, rather than x. Second order linear equations arise regularly in models of physical systems based on the application of Newton's 2nd Law, in which case the independent variable is time.

Solution: The auxiliary equation is $k^2 + 4k + 13 = 0$ which has complex roots $k = -2 \pm 3i$. The complementary function is therefore

$$y_c = e^{-2t}(A\cos 3t + B\sin 3t),$$

where A and B are arbitrary constants.
As a trial particular integral, try

$$\begin{aligned} y_p &= a\cos 3t + b\sin 3t, \\ \Rightarrow \quad \dot{y_p} &= -3a\sin 3t + 3b\cos 3t, \\ \Rightarrow \quad \ddot{y_p} &= -9a\cos 3t - 9b\sin 3t, \end{aligned}$$

where we have used the common notation

$$\dot{y_p} \equiv \frac{dy}{dt}, \quad \ddot{y_p} \equiv \frac{d^2y}{dt^2}.$$

Substituting into the ODE gives

$$(4a + 12b)\cos 3t + (-12a + 4b)\sin 3t = 20\cos 3t.$$

and equating the coefficients of $\cos 3t$ and $\sin 3t$ gives

$$\begin{aligned} 4a + 12b &= 20 \\ -12a + 4b &= 0. \end{aligned}$$

It follows that $a = \frac{1}{2}$, $b = \frac{3}{2}$. Therefore the general solution is

$$y(t) = y_p + y_c = \frac{1}{2}\cos 3t + \frac{3}{2}\sin 3t + e^{-2t}(A\cos 3t + B\sin 3t),$$

where A and B are arbitrary constants.

Note that second order non-homogeneous linear ODEs regularly arise as models of mechanical systems or electrical circuits. In these applications, the complementary functions is often referred to as the **transient** and the particular solution as the **forced solution**. Note that in the example above, the transient tends to zero as $t \to \infty$.

3.2.3 Initial and Boundary Conditions

We saw that the general solution of a first order equation contains a single arbirtrary constant, and that a single initial condition is required to determine the value of this constant. In contrast, the general solution of a second order equation contains two arbitrary constants.

Because of this, **two** conditions must be imposed to determine the values of these constants. These conditions can involve either y, its derivative(s), or both.
If the independent variable represents time, such conditions are typically referred to as **initial conditions**; if the independent variable represents space, they are typically referred to as **boundary conditions**. In practice, both are used in the same way: by specifying values for y and/or its derivative(s) at particular values of the independent variable, we can determine values for the arbitrary constants.

Example: Determine the solution of the ODE $\ddot{y} + 4\dot{y} + 13y = 20\cos 3t$ that satisfies the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$.

Solution: The general solution was derived in the previous example:

$$y(t) = \frac{1}{2}\cos 3t + \frac{3}{2}\sin 3t + e^{-2t}(A\cos 3t + B\sin 3t),$$

where A and B are arbitrary constants. Differentiating this expression gives

$$\dot{y}(t) = -\frac{3}{2}\sin 3t + \frac{9}{2}\cos 3t + e^{-2t}(-3A\sin 3t + 3B\cos 3t) - 2e^{-2t}(A\cos 3t + B\sin 3t).$$

So

$$y(0) = \frac{1}{2} + A, \quad \dot{y}(0) = \frac{9}{2} + 3B - 2A.$$

So

$$y(0) = 0 \quad \Rightarrow \quad A = -\frac{1}{2} \quad \text{and} \quad \dot{y}(0) = 1 \quad \Rightarrow \quad B = -\frac{3}{2}.$$

The required solution is therefore

$$y(t) = \frac{1}{2}\cos 3t + \frac{3}{2}\sin 3t - \frac{1}{2}e^{-2t}(\cos 3t + 3\sin 3t).$$

4 The Laplace Transform

The Laplace transform has many uses in Applied Mathematics, and we will see in this section how it can be used to solve linear ordinary differential equations. However, before discussing this particular application, we shall first give the basic definition of the Laplace transform, and derive some of its key properties.

4.1 Definition of the Laplace transform

The *Laplace transform* of a function $f(t)$ with domain $t > 0$ is defined by the integral

$$F(s) = \int_0^\infty e^{-st} f(t)\ \mathrm{d}t. \tag{4.1}$$

Note that the Laplace transform maps a *function* (of t) to another *function* (of s). We shall assume that s is real, but it may be a complex number (and, indeed, in some circumstances, it must be complex).

An alternative notation is

$$\mathcal{L}(f(t))(s) = F(s).$$

This notation has the advantage of making it clear that the original function was f. It is sometimes abbreviated to $\mathcal{L}(f)(s)$ or $\mathcal{L}(f(t))$ or $\mathcal{L}(f)$.

Example: Find the Laplace transform of $f(t) = t$.

Solution:

$$\begin{aligned}
\mathcal{L}(t) &= \int_0^\infty te^{-st}\,\mathrm{d}t \\
&= \left[t\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\cdot 1\,\mathrm{d}t \quad \text{(using integration by parts)} \\
&= \frac{1}{s}\int_0^\infty e^{-st}\,\mathrm{d}t \quad \text{(using } \lim_{t\to\infty} te^{-st} = 0 \text{ for } s > 0 \text{ — l'Hôpital's rule)} \\
&= \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^\infty \\
&= \frac{1}{s^2},
\end{aligned}$$

provided that $s > 0$. Note that it is common, as in this example, for the LT of a function to exist only for some restricted range of s.

4.2 Properties of the Laplace Transform

1. The Laplace Transform is a linear transformation. If a and b are constants, then

   $$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t)).$$

   This result is easily demonstrated by direct substitution into the definition of the LT.

2. **The shift theorem**. If $\mathcal{L}(f(t)) = F(s)$ and a is a constant, then

   $$\begin{aligned}
   \mathcal{L}\big(e^{at}f(t)\big) &= \int_0^\infty e^{-st}e^{at}f(t)\,\mathrm{d}t \\
   &= \int_0^\infty e^{-(s-a)t}f(t)\,\mathrm{d}t \\
   &= F(s-a).
   \end{aligned}$$

The shift theorem allows us to use standard LTs to evaluate other LTs. For example, since we know that $\mathcal{L}(t) = \dfrac{1}{s^2}$, then $\mathcal{L}(te^{3t}) = \dfrac{1}{(s-3)^2}$.

3. **Transforms of derivatives**. If $\mathcal{L}\big(f(t)\big) = F(s)$, then

$$\mathcal{L}\big(f'(t)\big) = sF(s) - f(0),$$

and

$$\mathcal{L}\big(f''(t)\big) = s^2F(s) - sf(0) - f'(0),$$

where $f'(t) = \dfrac{df}{dt}$ and $f''(t) = \dfrac{d^2f}{dt^2}$. To demonstrate this, we use integration by parts. For example,

$$\begin{aligned}\mathcal{L}\big(f'(t)\big) &= \int_0^\infty f'(t)e^{-st}\,\mathrm{d}t \\ &= \big[f(t)e^{-st}\big]_0^\infty + s\int_0^\infty e^{-st}f(t)\,\mathrm{d}t \\ &= f(0) + s\mathcal{L}\big(f(t)\big).\end{aligned}$$

Note that we have assumed that $\lim\limits_{t\to\infty} f(t)e^{-st} = 0$. The argument for the second derivative goes along the same lines, and we need to assume additionally that $\lim\limits_{t\to\infty} f'(t)e^{-st} = 0$.

4.3 Standard Laplace Transforms

All the following can be derived using integration by parts. a and ω are constants.

$f(t)$	$\mathcal{L}\big(f(t)\big) = F(s)$
t^n	$\dfrac{n!}{s^{n+1}} \quad (n = 0, 1, 2, \ldots)$
e^{at}	$\dfrac{1}{s-a}$
$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sinh\omega t$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh\omega t$	$\dfrac{s}{s^2-\omega^2}$

4.4 The Inverse Laplace Transform

When using LTs to solve a differential equation, we find an algebraic expression for the LT $F(s)$ of an unknown function ($f(t)$ — the dependent variable). The problem is then to find $f(t)$ given $F(s)$. Formally, this involves finding the *inverse Laplace transform*:

$$F(s) = \mathcal{L}\big(f(t)\big) \implies f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}.$$

Finding the LT of $f(t)$ is an exercise in integration; finding the inverse LT of $F(s)$ without prior knowledge is hard! What we do is to use a table of standard Laplace transforms, linearity, and rules like the shift rule, to deduce the inverse LT. For example, since we know that $\mathcal{L}(t) = \dfrac{1}{s^2}$, we can deduce that

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \quad (s > 0,\ t > 0).$$

Note that taking the inverse of the shift rule $\mathcal{L}\big(e^{at}f(t)\big) = F(s-a)$ gives

$$\mathcal{L}^{-1}\big(F(s-a)\big) = e^{at}f(t).$$

4.5 Examples

1. Find the LT of $f(t) = (1+t)^2e^{-2t}$.

$$\begin{aligned}
\mathcal{L}(f) &= \mathcal{L}\big((1+t)^2e^{-2t}\big) \\
&= \mathcal{L}\big((1+2t+t^2)e^{-2t}\big) \\
&= \mathcal{L}(e^{-2t}) + 2\mathcal{L}(te^{-2t}) + \mathcal{L}(t^2e^{-2t}) \quad \text{(using linearity)}
\end{aligned}$$

Now we can use the shift theorem, which tells us that

$$\begin{aligned}
\mathcal{L}(1) = \frac{1}{s} \quad &\implies \quad \mathcal{L}(e^{-2t}) = \frac{1}{s+2} \\
\mathcal{L}(t) = \frac{1}{s^2} \quad &\implies \quad \mathcal{L}(te^{-2t}) = \frac{1}{(s+2)^2} \\
\mathcal{L}(t^2) = \frac{2}{s^3} \quad &\implies \quad \mathcal{L}(t^2e^{-2t}) = \frac{2}{(s+2)^3}
\end{aligned}$$

Substituting into our previous expression gives

$$\mathcal{L}(f) = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{2}{(s+2)^3}.$$

2. Find $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+9}\right)$.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+9}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) = \frac{1}{3}\sin 3t,$$

where we have used linearity and then the table of standard LTs.

3. Find $\mathcal{L}^{-1}\left(\dfrac{1}{s^2-6s+5}\right)$.

We can evaluate this inverse LT in two ways. The first way relies on completing the square in the denominator and using the shift theorem for inverse LTs:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2-6s+5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-3)^2-4}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right) \\ &= \frac{e^{3t}}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2-4}\right) = \frac{1}{2}e^{3t}\sinh 2t. \end{aligned}$$

An alternative approach relies on factorising the denominator, using partial fractions, and the shift theorem for inverse LTs:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2-6s+5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-5)(s-1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{4}\cdot\frac{1}{s-5} - \frac{1}{4}\cdot\frac{1}{s-1}\right) \\ &= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = \frac{1}{4}\left(e^{5t}-e^{t}\right). \end{aligned}$$

Note that although the two answers may look superficially different, they are the same (using the standard definition of $\sinh$ ...)

4. Find $f(t) = \mathcal{L}^{-1}\left(\dfrac{4s}{s^2+6s+10}\right)$.

We first complete the square in the denominator:

$$f(t) = \mathcal{L}^{-1}\left(\frac{4s}{s^2+6s+10}\right) = \mathcal{L}^{-1}\left(\frac{4s}{(s+3)^2+1}\right).$$

Now the trick is to express the fraction as a function of the variable $s+3$, so that we can apply the shift theorem:

$$f(t) = \mathcal{L}^{-1}\left(\frac{4s}{(s+3)^2+1}\right) = \mathcal{L}^{-1}\left(\frac{4(s+3)-12}{(s+3)^2+1}\right) = e^{-3t}\mathcal{L}^{-1}\left(\frac{4s-12}{s^2+1}\right).$$

We then use linearity and the table of standard LTs:

$$\begin{aligned} f(t) = e^{-3t}\mathcal{L}^{-1}\left(\frac{4s-12}{s^2+1}\right) &= e^{-3t}\left[\mathcal{L}^{-1}\left(\frac{4s}{s^2+1}\right) - \mathcal{L}^{-1}\left(\frac{12}{s^2+1}\right)\right] \\ &= e^{-3t}\left[4\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) - 12\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)\right] \\ &= e^{-3t}\big(4\cos t - 12\sin t\big). \end{aligned}$$

4.6 Solving Linear ODEs using Laplace Transforms

If we take the Laplace transform of a linear ODE (with dependent variable $f(t)$), and use the results for derivatives:

$$\begin{aligned} \mathcal{L}\big(f'(t)\big) &= s\mathcal{L}\big(f(t)\big) - f(0) = sF(s) - f(0), \\ \mathcal{L}\big(f''(t)\big) &= s^2\mathcal{L}\big(f(t)\big) - sf(0) - f'(0) = s^2F(s) - sf(0) - f'(0), \end{aligned}$$

then we obtain an algebraic equation for $F(s) = \mathcal{L}\big(f(t)\big)$, which we can solve and then invert the LT to obtain $f(t)$. Note that since the transforms of derivatives involve initial conditions, we need to specify these in order to be able to solve for $f(t)$.

Example: Use Laplace transforms to solve the ODE

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0,$$

given that $y(0) = 1$ and $\dfrac{dy}{dt}(0) = 0$.

Solution: Let $\mathcal{L}\big(y(t)\big) = Y(s)$. Take the Laplace transform of the ODE, and use linearity:

$$\mathcal{L}\left(\frac{d^2y}{dt^2}\right) + 6\mathcal{L}\left(\frac{dy}{dt}\right) + 13\mathcal{L}(y) = 0.$$

Using the results for derivatives gives:

$$s^2Y(s) - sy(0) - y'(0) + 6\big[sY(s) - y(0)\big] + 13Y(s) = 0.$$

Applying the initial conditions and rearranging gives

$$Y(s) = \frac{s+6}{s^2+6s+13}.$$

We can now find $y(t)$ by inverting the Laplace transform:

$$y(t) = \mathcal{L}^{-1}\big(Y(s)\big) = \mathcal{L}^{-1}\left(\frac{s+6}{s^2+6s+13}\right).$$

We proceed by completing the square in the denominator:

$$y(t) = \mathcal{L}^{-1}\left(\frac{s+6}{(s+3)^2+4}\right) = \mathcal{L}^{-1}\left(\frac{(s+3)+3}{(s+3)^2+4}\right),$$

where we have expressed the numerator in terms of the variable $s+3$, so that we can apply the shift theorem:

$$y(t) = \mathcal{L}^{-1}\left(\frac{(s+3)+3}{(s+3)^2+4}\right) = e^{-3t}\mathcal{L}^{-1}\left(\frac{s+3}{s^2+4}\right) = e^{-3t}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}+\frac{3}{s^2+4}\right)$$

Finally, we can apply linearity and consult the table of standard transforms to get

$$y(t) = e^{-3t}\left[\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)+\frac{3}{2}\,\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right)\right] = e^{-3t}\left(\cos 2t+\frac{3}{2}\sin 2t\right).$$

Exercise: Check that you get the same result using the techniques of the previous section on ODEs.

END OF NOTES