University of Sheffield

School of Mathematics and Statistics

MAS140: Mathematics (Chemical)
MAS152: Civil Engineering Mathematics
MAS152: Essential Mathematical Skills & Techniques
MAS156: Mathematics (Electrical and Aerospace)
MAS161: General Engineering Mathematics

Semester 1 2017–18

Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,
`http://engmaths.group.shef.ac.uk/mas140/`
`http://engmaths.group.shef.ac.uk/mas151/`
`http://engmaths.group.shef.ac.uk/mas152/`
`http://engmaths.group.shef.ac.uk/mas156/`
`http://engmaths.group.shef.ac.uk/mas161/`
which can also be accessed through MOLE.

1 Vectors

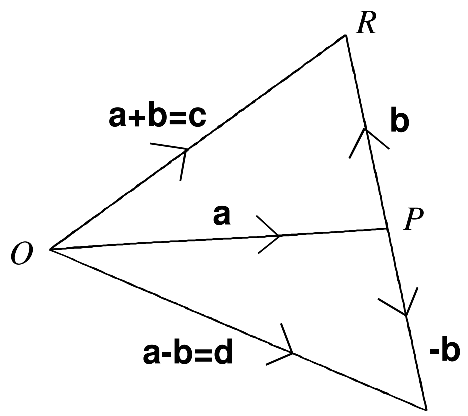
A $\boxed{\text{scalar}}$ is a quantity which has magnitude but not direction in space.
For example speed, volume, energy are all scalar quantities.
A $\boxed{\text{vector}}$ has magnitude and a definite direction in space.
For example velocity, force, magnetic field are all examples of vectors.
Notation. In books and examination papers, vectors are written in **bold** type. So that **a** and **A** are vectors.
In hand-written work the usual way to indicate a vector is by $\underset{\sim}{a}$. Thus y is a scalar but $\underset{\sim}{x}$ is a vector.
The $\boxed{\text{magnitude}}$ (or $\boxed{\text{modulus}}$)of the vector **a** or $\underset{\sim}{a}$ is denoted by a or $|\mathbf{a}|$ or $|\underset{\sim}{a}|$.
Note that

$$|\mathbf{a}| \geq 0.$$

A $\boxed{\text{line segment}}$ also has magnitude (or length) and direction and is a very convenient way of representing a vector.

1.1 Addition and Subtraction of Vectors

Vectors are added by using the triangle law which operates as follows:



To add two vectors **a** and **b**, we take a fixed origin O and take $\overrightarrow{OP} = \mathbf{a}$ (i.e. the length $OP = a$ and $\overrightarrow{OP}$ is in the same direction as **a**; thus $\overrightarrow{OP}$ is parallel to **a**). Similarly find the point R so that $\overrightarrow{PR} = \mathbf{b}$. Then $\overrightarrow{OR} = \mathbf{a} + \mathbf{b} = \mathbf{c}$.
The vector $-\mathbf{b}$ is defined to be the vector of the same length (i.e. magnitude) as **b** but in the opposite direction.
We define $\mathbf{a} - \mathbf{b}$ to be $\mathbf{a} + (-\mathbf{b})$. So

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

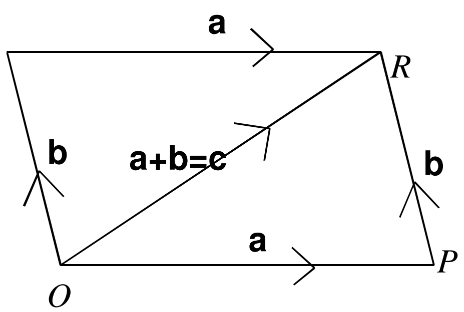
or we add the vector $-\mathbf{b}$ to the vector **a**. The resultant is $\mathbf{d} = \mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$.
From the diagram it can also be seen that

$$\mathbf{b} - \mathbf{b} = \mathbf{b} + (-\mathbf{b}) = \mathbf{0}.$$

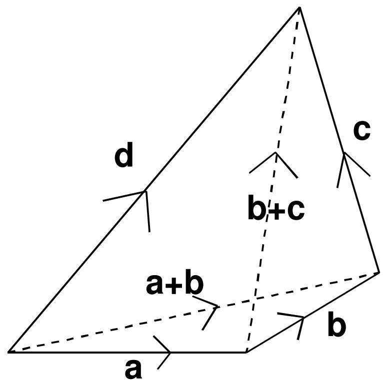
Do not confuse $\mathbf{0} = \underset{\sim}{0}$ with 0.

Note that:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$



2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) = \mathbf{d}$



Since it does not matter which pair we combine first, we write $\mathbf{d} = \mathbf{a} + \mathbf{b} + \mathbf{c}$ without any brackets at all.

1.2 Multiplication by a scalar

The product of a vector $\mathbf{a} = \underset{\sim}{a}$ and a scalar m is denoted by $m\mathbf{a}$ or $\mathbf{a}m$.

If m is positive then $m\mathbf{a}$ is parallel to $\mathbf{a}$ and its length is $m|\mathbf{a}|$.
If m is negative then $m\mathbf{a}$ is anti-parallel to $\mathbf{a}$ (i.e. it is in the opposite direction to $\mathbf{a}$) and its length is $|m||\mathbf{a}| = -m|\mathbf{a}|$.

1.3 Unit Vectors

A $\boxed{\text{unit vector}}$ is a vector whose magnitude (or length) is unity.
The unit vector parallel to $\mathbf{a}$ is denoted by $\hat{\mathbf{a}}$. Clearly

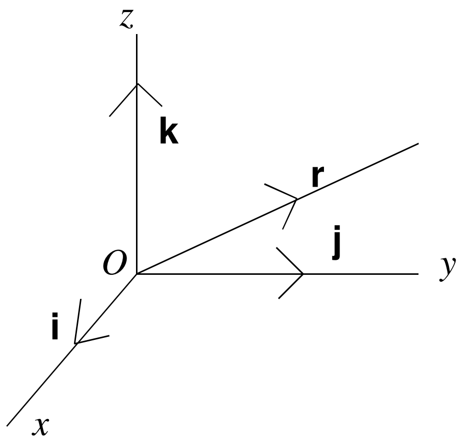
$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{a} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

because this vector is parallel to $\mathbf{a}$ and

$$|\hat{\mathbf{a}}| = \left|\frac{\mathbf{a}}{a}\right| = \frac{|\mathbf{a}|}{a} = \frac{a}{a} = 1.$$

1.4 Rectangular Resolution of Vectors

We take an origin O and a set of 3 fixed axes Ox, Oy and Oz which are mutually perpendicular. The axes Ox, Oy and Oz *in that order* form a right-handed orthogonal system.



The Oy and Oz axes are in the plane of the paper and Ox is *outwards*. We also take fixed $\boxed{\text{unit}}$ vectors $\mathbf{i} = \underset{\sim}{i}$, $\mathbf{j}$ and $\mathbf{k}$ along Ox, Oy and Oz respectively.

Suppose that $\overrightarrow{OP}$ represents the vector $\mathbf{r}$ where P has $\boxed{\text{coordinates}}$ (x, y, z). Then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and the vector $\mathbf{r}$ is $\boxed{\text{resolved}}$ into three $\boxed{\text{components}}$ parallel to the three axes.
We will write

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (x, y, z).$$

This means that the components of $\mathbf{r}$ parallel to the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are x, y and z respectively.
Two vectors are equal when all three of their respective components are equal.
So $a\mathbf{i} + b\mathbf{j} + c\mathbf{k} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ implies (and is implied by) $x = a$, $y = b$ and $z = c$.

Examples

1. If $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$ then $a_1 = -1$, $a_2 = 2$ and $a_3 = 5$.
2. Express the vector $\mathbf{a} = (4, 1, 0)$ as the sum of two vectors, one parallel to the vector $\mathbf{b} = (1, 2, 0)$ and the other parallel to the vector $\mathbf{c} = (4, -6, 0)$.

Solution: We want scalars β and γ so that

$$\mathbf{a} = \beta\mathbf{b} + \gamma\mathbf{c}.$$

So $(4, 1, 0) = \beta(1, 2, 0) + \gamma(4, -6, 0)$ which implies

$$4 = \beta + 4\gamma \qquad (1.1)$$
$$1 = 2\beta - 6\gamma \qquad (1.2)$$

Multiply (1) by 2 to get

$$8 \;=\; 2\beta + 8\gamma. \tag{3}$$

and subtract (2) from (3) to give

$$7 = 14\gamma \;\Rightarrow\; \gamma = 1/2.$$

(1) now gives $\beta = 2$ and so

$$\mathbf{a} = 2\mathbf{b} + \frac{1}{2}\mathbf{c}.$$

3. The vectors **a** and **b** defined by

$$\mathbf{a} = (3, -1, 4) \;\text{ and }\; \mathbf{b} = (2, \lambda, \mu)$$

are parallel. Find the scalars λ and μ.

Solution: Since **a** is parallel to **b**, then
$\mathbf{a} = m\mathbf{b}$, for some scalar m. Hence

$$(3, -1, 4) = m(2, \lambda, \mu)$$

giving

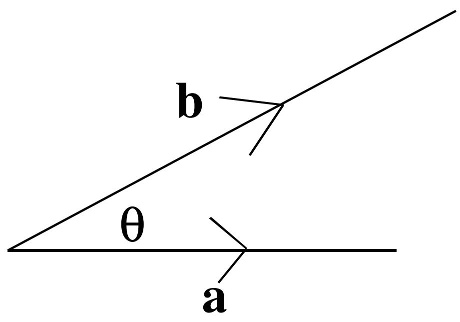
$$3 = 2m, \;\; -1 = \lambda m \text{ and } 4 = \mu m.$$

So $m = 3/2$ and $\lambda = -1/m = -2/3$
and $\mu = 4/m = 8/3$.

1.5 The Scalar Product of Two Vectors

Let the angle between the two vectors **a** and **b** be θ. We write the $\boxed{\text{scalar}}$ product of **a** and **b** as **a**.**b** and define it to be

$$\mathbf{a}.\mathbf{b} = ab\cos\theta.$$



The scalar product produces a $\boxed{\text{scalar}}$ from the two $\boxed{\text{vectors}}$. It is also known as the $\boxed{\text{dot}}$ product. Note that

$$\mathbf{b}.\mathbf{a} = ba\cos\theta = ab\cos\theta = \mathbf{a}.\mathbf{b}$$

If **a** is perpendicular to **b** then $\theta = \pi/2$ and $\cos\theta = 0$. So $\mathbf{a}.\mathbf{b} = 0$. Also

$$\mathbf{a}.\mathbf{a} = aa\cos 0 = a^2 = |\mathbf{a}|^2.$$

In particular

$$\mathbf{i}.\mathbf{i} = 1 = \mathbf{j}.\mathbf{j} = \mathbf{k}.\mathbf{k}$$

and

$$\mathbf{i}.\mathbf{j} = 0 = \mathbf{j}.\mathbf{k} = \mathbf{k}.\mathbf{i}$$

Thus, if

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \end{aligned}$$

then

$$\begin{aligned} \mathbf{a}.\mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}).(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i}.\mathbf{i} + a_1b_2\mathbf{i}.\mathbf{j} + a_1b_3\mathbf{i}.\mathbf{k} \\ &\quad + a_2b_1\mathbf{j}.\mathbf{i} + a_2b_2\mathbf{j}.\mathbf{j} + a_2b_3\mathbf{j}.\mathbf{k} \\ &\quad + a_3b_1\mathbf{k}.\mathbf{i} + a_3b_2\mathbf{k}.\mathbf{j} + a_3b_3\mathbf{k}.\mathbf{k} \\ &= a_1b_1 + 0 + 0 \\ &\quad + 0 + a_2b_2 + 0 \\ &\quad + 0 + 0 + a_3b_3 \end{aligned}$$

Thus

$$\boxed{\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

It follows that $\mathbf{a}.\mathbf{a} = a_1^2 + a_2^2 + a_3^2$ or

$$a = |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Example 1. Find the unit vector $\hat{\mathbf{a}}$ which is parallel to $\mathbf{a} = (a_1, a_2, a_3)$.

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{a} = \frac{(a_1, a_2, a_3)}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

Example 2. Find the angle between the vectors

$$\begin{aligned} \mathbf{a} &= 3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \\ \mathbf{b} &= -\mathbf{i} + \mathbf{j} + 3\mathbf{k} \end{aligned}$$

Solution. Let θ be the angle between the two vectors. Then

$$ab\cos\theta = \mathbf{a}.\mathbf{b} = -3 + 4 + 3 = 4$$

Also $a = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$
and $b = \sqrt{1 + 1 + 9} = \sqrt{11}$. Hence

$$\sqrt{26}\sqrt{11}\cos\theta = 4 \quad \Rightarrow \quad \cos\theta = \frac{4}{\sqrt{286}}$$

1.6 The Vector Product of Two Vectors

Let θ be the angle between the vectors **a** and **b**. The $\boxed{\text{vector}}$ product of these two vectors is written as $\mathbf{a} \times \mathbf{b}$ and is defined by

$$\mathbf{a} \times \mathbf{b} = ab \sin\theta \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector which is perpendicular to both **a** and **b** and its direction is given by the right-hand-screw rule.
A right-hand screw directed along $\hat{\mathbf{n}}$ rotates from **a** to **b** through the (acute) angle θ.
Note that

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

Because of the notation used, the vector product is also known as the $\boxed{\text{cross}}$ product.

If **a** and **b** are parallel (antiparallel) then $\theta = 0$ $(\theta = \pi)$ and so $\sin\theta = 0$.
Consequently $\mathbf{a} \times \mathbf{b} = \mathbf{0}$
Also, if we know that $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then *either* $a = 0$ *or* $b = 0$ *or* **a** and **b** are parallel (or antiparallel). It is common to use *parallel* to include both parallel and antiparallel.
We see that

$$\mathbf{i} \times \mathbf{i} = \mathbf{0} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k}$$

and

$$\begin{array}{ll} \mathbf{i} \times \mathbf{j} = \mathbf{k}, & \mathbf{j} \times \mathbf{i} = -\mathbf{k} \\ \mathbf{j} \times \mathbf{k} = \mathbf{i}, & \mathbf{k} \times \mathbf{j} = -\mathbf{i} \\ \mathbf{k} \times \mathbf{i} = \mathbf{j}, & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

The cross product of any two of **i**, **j**, **k** is the third provided that the order **i j k i j** ... is maintained. If the order is reversed then the negative is obtained.
If

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \\ \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} \end{aligned}$$

then

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i} \times \mathbf{i} + a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} \\ &\quad + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_2\mathbf{j} \times \mathbf{j} + a_2b_3\mathbf{j} \times \mathbf{k} \\ &\quad + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} + a_3b_3\mathbf{k} \times \mathbf{k} \\ &= a_1b_2\mathbf{k} - a_1b_3\mathbf{j} \\ &\quad - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} \\ &\quad + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} \\ &= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} \\ &\quad + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \end{aligned}$$

Using the notation of $\boxed{\text{determinants}}$, we can write

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

1.7 Determinants

$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is called a determinant of $\boxed{\text{order 2}}$ and its value is $a_1b_2 - a_2b_1$. So

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

For example,

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \times 4 - 2 \times 3 = -2.$$

Also

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

is a determinant of $\boxed{\text{order 3}}$. Its value is

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Note the minus sign!

In general, a determinant of order n consists of n rows and n columns. Its numerical value is obtained by extending the ideas above. In the context of vectors, we shall only meet second and third order determinants (i.e. $n = 2$ and $n = 3$). However, we shall have more to say about determinants in the second part of this module.

Example:

$$\begin{aligned} \begin{vmatrix} 1 & 3 & -2 \\ 2 & -1 & 0 \\ -3 & 2 & 5 \end{vmatrix} &= 1 \begin{vmatrix} -1 & 0 \\ 2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 2 & 0 \\ -3 & 5 \end{vmatrix} \\ &\quad - 2 \begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} \\ &= 1(-5 - 0) - 3(10 - 0) \\ &\quad - 2(4 - 3) \\ &= -5 - 30 - 2 = -37 \end{aligned}$$

Using determinants, we see that the vector product $\mathbf{a} \times \mathbf{b}$ can be written as

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)
\end{aligned}$$

Example 1. If $\mathbf{a} = (1, -2, 3)$ and $\mathbf{b} = (3, 0, -1)$ then

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ 3 & 0 & -1 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -2 & 3 \\ 0 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} \\
&= 2\mathbf{i} + 10\mathbf{j} + 6\mathbf{k}
\end{aligned}$$

Note that

$$\begin{aligned}
\mathbf{a}.(2, 10, 6) &= 2 - 20 + 18 = 0 \\
\mathbf{b}.(2, 10, 6) &= 6 + 0 - 6 = 0
\end{aligned}$$

Always perform this check when finding cross products.

Example 2. Find the constants λ and μ so that the vectors $(\lambda, 4, \mu)$ and $(3, 2, -1)$ are parallel.

Solution. If these vectors are parallel then their vector product is the zero vector. So

$$\mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \lambda & 4 & \mu \\ 3 & 2 & -1 \end{vmatrix}$$

$$= \mathbf{i}\begin{vmatrix} 4 & \mu \\ 2 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} \lambda & \mu \\ 3 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} \lambda & 4 \\ 3 & 2 \end{vmatrix}$$

$$= \mathbf{i}(-4 - 2\mu) - \mathbf{j}(-\lambda - 3\mu) + \mathbf{k}(2\lambda - 12)$$

and so

$$\begin{aligned}
-4 - 2\mu &= 0 \\
\lambda + 3\mu &= 0 \\
2\lambda - 12 &= 0
\end{aligned}$$

All 3 of these equations are satisfied by $\lambda = 6$ and $\mu = -2$.

Example 3. Verify the identity

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{c}.\mathbf{a})\mathbf{b}$$

for the vectors $\mathbf{a} = (1,2,3)$, $\mathbf{b} = (2,0,-1)$ and $\mathbf{c} = (4,-1,-1)$.

Solution.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 2 & 0 & -1 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \\
&= -2\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & -1 \\ -2 & 7 & -4 \end{vmatrix} \\
&= \mathbf{i} \begin{vmatrix} -1 & -1 \\ 7 & -4 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & -1 \\ -2 & -4 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & -1 \\ -2 & 7 \end{vmatrix} \\
&= 11\mathbf{i} + 18\mathbf{j} + 26\mathbf{k}
\end{aligned}$$

$$\mathbf{c}.\mathbf{b} = (4,-1,-1).(2,0,-1) = 8 + 0 + 1 = 9$$
$$\mathbf{c}.\mathbf{a} = (4,-1,-1).(1,2,3) = 4 - 2 - 3 = -1$$

$$\begin{aligned}
(\mathbf{c}.\mathbf{b})\mathbf{a} - (\mathbf{c}.\mathbf{a})\mathbf{b} &= 9\mathbf{a} + \mathbf{b} \\
&= (9+2, 18+0, 27-1) \\
&= (11, 18, 26) \\
&= \mathbf{c} \times (\mathbf{a} \times \mathbf{b})
\end{aligned}$$

1.8 Scalar Triple Product

The quantity $\mathbf{a}.(\mathbf{b} \times \mathbf{c})$ is known as a scalar triple product. (Sometimes written as $\mathbf{a}.\mathbf{b} \times \mathbf{c}$ i.e. without brackets.)
It can be shown that if

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$
$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$
$$\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$$

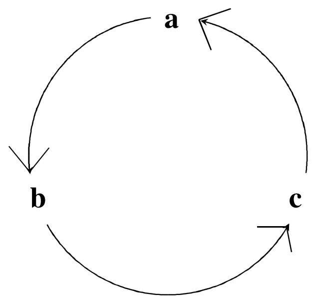
then

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

It can also be shown that

$$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}).\mathbf{c} = (\mathbf{c} \times \mathbf{a}).\mathbf{b}$$

So the "dot" and "cross" can be interchanged provided that the cyclic order of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is maintained.



If the order is changed, then the sign is changed. So

$$(\mathbf{a} \times \mathbf{b}).\mathbf{c} = -(\mathbf{b} \times \mathbf{a}).\mathbf{c}$$

Example. Prove that $\mathbf{a}.(\mathbf{b} \times \mathbf{a}) = 0$.

Solution.

$\mathbf{a}.(\mathbf{b} \times \mathbf{a}) = -\mathbf{a}.(\mathbf{a} \times \mathbf{b}) = -(\mathbf{a} \times \mathbf{a}).\mathbf{b} = 0$
(because $\mathbf{a} \times \mathbf{a} = \mathbf{0}$).

1.9 Differentiation of Vectors

A vector may be considered as a function of one or more independent variables. For example, the velocity of an accelerating body is a function of time, and the local wind velocity in the atmosphere is a function of both time and position. If a vector is a function of one or more independent variables, then we can consider its derivatives with respect to these. We can work by applying the standard principles of differentiation to each component of the vector independently. We will focus here on time-dependent vectors.

1.9.1 Rules for Differentiation

1. If $\mathbf{c}$ is a $\boxed{\text{constant}}$ vector (a vector whose magnitude and direction do not vary with time t) then
$$\frac{d\mathbf{c}}{dt} = \mathbf{0}$$
In particular
$$\frac{d\mathbf{i}}{dt} = \mathbf{0}, \quad \frac{d\mathbf{j}}{dt} = \mathbf{0}, \quad \frac{d\mathbf{k}}{dt} = \mathbf{0}$$
2. If the vector $\mathbf{r}$ and the scalar m both depend upon t then
$$\frac{d}{dt}(m\mathbf{r}) = \frac{dm}{dt}\mathbf{r} + m\frac{d\mathbf{r}}{dt}$$
3. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where x, y and z are functions of t, then
$$\frac{d\mathbf{r}}{dt} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}$$

or

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$$

4. If $\mathbf{r}_1$ and $\mathbf{r}_2$ are vector functions of t then

$$\frac{d}{dt}(\mathbf{r}_1.\mathbf{r}_2) = \frac{d\mathbf{r}_1}{dt}.\mathbf{r}_2 + \mathbf{r}_1.\frac{d\mathbf{r}_2}{dt}$$

and

$$\frac{d}{dt}(\mathbf{r}_1 \times \mathbf{r}_2) = \frac{d\mathbf{r}_1}{dt} \times \mathbf{r}_2 + \mathbf{r}_1 \times \frac{d\mathbf{r}_2}{dt}$$

In particular, $\frac{d}{dt}(\mathbf{r}.\mathbf{r}) = \frac{d\mathbf{r}}{dt}.\mathbf{r} + \mathbf{r}.\frac{d\mathbf{r}}{dt} = 2\mathbf{r}.\frac{d\mathbf{r}}{dt}$

Now $\mathbf{r}.\mathbf{r} = |\mathbf{r}|^2 = r^2$ and so $\mathbf{r}.\frac{d\mathbf{r}}{dt} = r\frac{dr}{dt}$

If it is known that $\mathbf{r}$ has constant magnitude, then $\dot{r} = 0$ and so $\mathbf{r}$ is perpendicular to $\dot{\mathbf{r}}$.

Example. Let $\mathbf{r} = \cos\omega t\mathbf{i} + \sin\omega t\mathbf{j}$ where ω is a constant. Then

$$\frac{d\mathbf{r}}{dt} = -\omega\sin\omega t\mathbf{i} + \omega\cos\omega t\mathbf{j}$$

1.9.2 Application to Dynamics

It is very common in dynamics to use dots to denote differentiation with respect to t.

If $\mathbf{r}$ is the position vector of a particle with respect to some fixed origin O, then

$$\frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} = \mathbf{v}$$

is the $\boxed{\text{velocity}}$ of the particle. The $\boxed{\text{speed}}$ of the particle is the scalar

$$v = |\mathbf{v}| = |\dot{\mathbf{r}}|$$

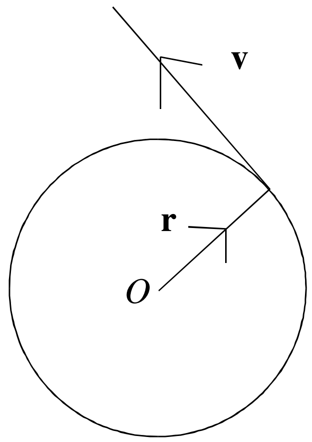
The $\boxed{\text{acceleration}}$ of the particle is the vector

$$\mathbf{f} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2}$$

or

$$\mathbf{f} = \dot{\mathbf{v}} = \ddot{\mathbf{r}}$$

For a particle moving in a circle of radius a, $|\mathbf{r}| = r = a$ and so $\dot{r} = 0$. Hence the velocity of the particle is perpendicular to $\mathbf{r}$ (see above), that is, it is tangential to the circle.



Example. Show that $\dfrac{d}{dt}\left(\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}\right)=\mathbf{r}\times\dfrac{d^2\mathbf{r}}{dt^2}$

Solution

$$\begin{aligned}\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) &= \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\\ &= \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\end{aligned}$$

because $\dfrac{d\mathbf{r}}{dt}\times\dfrac{d\mathbf{r}}{dt}=\mathbf{0}$

Example. Let $\mathbf{r}=b(\cos\omega t-2)\mathbf{i}+b\sin\omega t\mathbf{j}+c\omega t\mathbf{k}$, where b, c and ω are positive constants. Find the velocity, speed and acceleration of the particle at time t. When is the acceleration perpendicular to the position vector?

Solution. The velocity is $\mathbf{v}$ where

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=-\omega a\sin\omega t\mathbf{i}+a\omega\cos\omega t\mathbf{j}+c\omega\mathbf{k}$$

and the speed is v where

$$\begin{aligned}v=|\mathbf{v}| &= \sqrt{\mathbf{v}.\mathbf{v}}\\ &= \sqrt{b^2\omega^2\sin^2\omega t+b^2\omega^2\cos^2\omega t+c^2\omega^2}\\ &= \sqrt{b^2\omega^2+c^2\omega^2}=\omega\sqrt{b^2+c^2}\end{aligned}$$

The acceleration is $\mathbf{a}$ where

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=-b\omega^2\cos\omega t\mathbf{i}-b\omega^2\sin\omega t\mathbf{j}$$

The acceleration and the position vector are perpendicular when $\mathbf{a}.\mathbf{r}=0$.
Now

$$\begin{aligned}\mathbf{a}.\mathbf{r} &= -b\omega^2\cos\omega t(b\cos\omega t-2b)-b^2\omega^2\sin^2\omega t\\ &= -b^2\omega^2(\cos^2\omega t-2\cos\omega t+\sin^2\omega t)\\ &= b^2\omega^2(2\cos\omega t-1)\end{aligned}$$

Thus $\mathbf{a}.\mathbf{r}=0$ when

$$2\cos\omega t=1\quad\Rightarrow\quad\cos\omega t=1/2.$$

Hence **a** is perpendicular to **r** at times given by

$$\omega t = \pm\frac{\pi}{3} + 2n\pi \quad (n = 0, \pm 1, \pm 2, \dots)$$

The position vector **r** describes helical motion in the z-direction, centred on the line $x = 2b, y = 0$. The radius of the circular motion is b. The figure shows an example for $b = 2, \omega = 1$ and $0 \le t \le 20$.