University of Sheffield

School of Mathematics and Statistics

MAS140: Mathematics (Chemical)
MAS152: Civil Engineering Mathematics
MAS152: Essential Mathematical Skills & Techniques
MAS156: Mathematics (Electrical and Aerospace)
MAS161: General Engineering Mathematics

Semester 1 2016–17

Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,
`http://engmaths.group.shef.ac.uk/mas140/`
`http://engmaths.group.shef.ac.uk/mas151/`
`http://engmaths.group.shef.ac.uk/mas152/`
`http://engmaths.group.shef.ac.uk/mas156/`
`http://engmaths.group.shef.ac.uk/mas161/`
which can also be accessed through MOLE.

1 Complex Numbers

The equation

$$x^2 - 1 = 0$$

has the solutions $x = \pm 1$ and the equation

$$\begin{aligned} x^2 - 6x + 5 &= 0 \\ \Rightarrow (x-5)(x-1) &= 0 \end{aligned}$$

has the roots 5 and 1. However, the equation

$$x^2 + 1 = 0$$

is not satisfied by any (real) number x (indeed, $x^2 \geq 0$ for all $x \in \mathbb{R}$).
We define the quantity i by $i = \sqrt{-1}$ so that $i^2 = -1$. The last equation now has the two roots i and $-i$. (Do $\boxed{\text{not}}$ be tempted to think that one of them is positive and one negative. The quantities i and $-i$ are neither positive nor negative!)
A quantity z of the form

$$z = x + iy,$$

where x and y are real numbers, is called a $\boxed{\text{complex}}$ number. They occur naturally when solving quadratic equations. Thus if

$$25z^2 - 20z + 13 = 0$$

then (using the standard formula)

$$\begin{aligned} z &= \frac{20 \pm \sqrt{20^2 - 4 \times 25 \times 13}}{2 \times 25} \\ &= \frac{2 \pm \sqrt{-9}}{5} \\ &= \frac{2}{5} \pm i\frac{3}{5} \end{aligned}$$

For any real numbers x and y, $z = x + iy$ is a complex number. In particular, choosing y to be zero shows that all real numbers are complex. In the same way, all integers are rational.

$$\mathbb{C} \supset \mathbb{R} \supset \mathbb{Q} \supset \mathbb{Z}$$

We do not distinguish between any of the following:

$$x + iy, \quad x + yi, \quad iy + x, \quad yi + x.$$

The symbol j is often used for $\sqrt{-1}$, especially in electrical engineering.
In what follows, x and y will always be $\boxed{\text{real}}$ numbers and z a $\boxed{\text{complex}}$ number.

If $z = x + iy$ then

$$x \text{ is called the } \boxed{\text{real part}} \text{ of } z$$
$$y \text{ is called the } \boxed{\text{imaginary part}} \text{ of } z$$

and we write

$$x = Re(z) \text{ and } y = Im(z).$$

Note. The imaginary part of a complex number is $\boxed{\text{real}}$.

Example
If $z = 4 - 3i$ then $Re(z) = 4$ and $Im(z) = -3$.
($\Re(z)$ and $\Im(z)$ are also used.)

A complex number of the form iy is called $\boxed{\text{pure-imaginary}}$. Its real part is zero.
The $\boxed{\text{complex conjugate}}$ of the complex number $z = x + iy$ is denoted by $\bar{z}$ and is defined by $\bar{z} = x - iy$.
Example $z = 4 - 3i \Rightarrow \bar{z} = 4 + 3i$.

Taking the complex conjugate twice gives the original number. i.e. if $z_1 = \bar{z}$, then $\bar{z}_1 = z$.
If $z = \bar{z}$ then $Im(z) = 0$ and z is a real number.

1.1 Algebraic Operations

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

- **Equality**

$$z_1 = z_2 \quad \Leftrightarrow \quad x_1 = x_2 \text{ and } y_1 = y_2$$

- **Addition**

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

- **Subtraction**

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

- **Multiplication**

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \end{aligned}$$

- **Division**

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \frac{(x_1 + iy_1)}{(x_2 + iy_2)} \frac{(x_2 - iy_2)}{(x_2 - iy_2)} \\ &= \frac{(x_1 x_2 + y_1 y_2)}{(x_2^2 + y_2^2)} + i \frac{(-x_1 y_2 + y_1 x_2)}{(x_2^2 + y_2^2)} \end{aligned}$$

Example: Find the real and imaginary parts of $z = \dfrac{3-4i}{5+2i}$

Solution

$$\begin{aligned}\frac{3-4i}{5+2i} &= \frac{(3-4i)}{(5+2i)}\frac{(5-2i)}{(5-2i)} \\ &= \frac{(3\times 5-4\times 2)+i(-3\times 2-4\times 5)}{(25+4)} \\ &= \frac{7-26i}{29}\end{aligned}$$

The real part is $Re(z) = 7/29$ and the imaginary part is $Im(z) = -26/29$.

For any complex numbers z_1, z_2 and z_3 we have

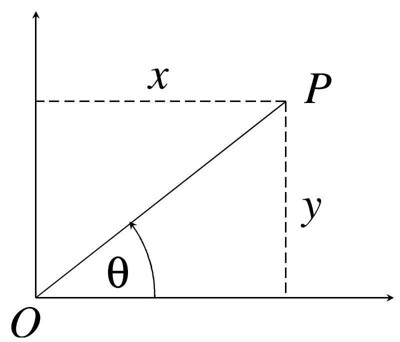
$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 && \text{(commutative)} \\ z_1 z_2 &= z_2 z_1 \\ z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 && \text{(associative)} \\ z_1(z_2 z_3) &= (z_1 z_2) z_3 \end{aligned}$$

Note

- $z + \bar{z} = 2Re(z)$ and $z - \bar{z} = 2iIm(z)$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- $\overline{z_1 z_2} = (\overline{z_1})(\overline{z_2})$, $\overline{z_1/z_2} = \overline{z_1}/\overline{z_2}$

1.2 Geometrical Representation: The Argand Diagram

The complex number $z = x+iy$ can be represented by the point P with coordinates (x, y) in a plane. In this plane, known as the $\boxed{\text{Argand Diagram}}$, the x-axis is called the real axis and the y-axis is the imaginary axis.



1.2.1 Modulus and Argument

Instead of using Cartesian coordinates (i.e. x and y) to specify the position of P in the Argand diagram, we may use $\boxed{\text{polar}}$ coordinates r and θ. Then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

and so

$$\boxed{z = x + iy = r(\cos\theta + i\sin\theta).}$$

This is called the $\boxed{\text{polar}}$ form of the complex number z. The non-negative real number r is the $\boxed{\text{modulus}}$ of z and θ is the $\boxed{\text{argument}}$ of z.
We use the notation

$$|z| = r \quad \text{and} \quad \arg z = \theta.$$

Clearly

$$r = \sqrt{x^2 + y^2}, \ \ \sin\theta = y/r \text{ and } \cos\theta = x/r.$$

The angle θ is measured positive in the anti-clockwise direction from the real axis. Values of θ which differ by 2π (or 4π etc) correspond to the same direction in the Argand diagram. The unique angle θ such that $-\pi < \theta \leq \pi$ is called the $\boxed{\text{principal}}$ value of the argument and we shall denote it by Arg$z$.

Example: Find the modulus and principal argument of

(*a*) $1 + i\sqrt{3}$ (*b*) $-1 + i$

Solution

(*a*) Let $z = 1 + i\sqrt{3}$, then

$|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$ and θ is such that

$$\cos\theta = \frac{1}{2}, \ \ \sin\theta = \frac{\sqrt{3}}{2} \text{ and } -\pi < \theta \leq \pi.$$

Hence the principal value of the argument is $\pi/3$.

(*b*) $|-1 + i| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$ and θ is such that

$$\cos\theta = -\frac{1}{\sqrt{2}}, \ \ \sin\theta = \frac{1}{\sqrt{2}} \text{ and } -\pi < \theta \leq \pi.$$

Hence the principal value of the argument is $3\pi/4$.

Note

(*a*) $|z| = |\bar{z}|$

(*b*) $z = x + iy \Rightarrow z\bar{z} = |z|^2 = x^2 + y^2$

(*c*) $Re(z) \leq |z|$ and $Im(z) \leq |z|$

It follows from (*b*) that

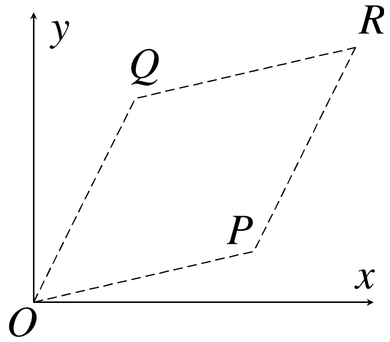
$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

In particular, if $|z| = 1$ then $1/z = \bar{z}$.
If the complex number z has modulus r and argument θ, then this is sometimes written as $z = \langle r, \theta \rangle$. Also

$$z = \langle r, \theta \rangle \Rightarrow \bar{z} = \langle r, -\theta \rangle$$

1.2.2 Addition and the Argand Diagram

If P represents z_1 and Q represents z_2 then R will represent their sum $z_1 + z_2$. Note that $OPRQ$ is a parallelogram. Also the triangle inequality gives



$$OP + PR \geq OR$$

and hence

$$OP + OQ \geq OR.$$

This implies that

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

1.3 Multiplication of Complex Numbers in Polar Form

Let

$$\begin{aligned} z_1 &= r_1(\cos\theta_1 + i\sin\theta_1) \\ \text{and} \quad z_2 &= r_2(\cos\theta_2 + i\sin\theta_2) \end{aligned}$$

then

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \\ &\qquad i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2[(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

Or

$$\langle r_1, \theta_1 \rangle \langle r_2, \theta_2 \rangle = \langle r_1 r_2, \theta_1 + \theta_2 \rangle$$

Also

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\ &= \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2) \\ &= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \\ &\qquad i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)] \\ &= \frac{r_1}{r_2}[(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \end{aligned}$$

Or

$$\frac{\langle r_1, \theta_1 \rangle}{\langle r_2, \theta_2 \rangle} = \left\langle \frac{r_1}{r_2}, \theta_1 - \theta_2 \right\rangle$$

Hence we have the following results for multiplication and division:

(*a*) $|z_1 z_2| = |z_1||z_2|$

(*b*) $|z_1/z_2| = |z_1|/|z_2|$

(*c*) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(*d*) $\arg(z_1/z_2) = \arg z_1 - \arg z_2$

Note that it is not possible to use Arg in results (c) and (d).

1.4 Loci

If the complex number z satisfies the equation

$$|z| = 1$$

then its distance from the origin is 1. So z lies on the circle of radius 1 centred on the origin. This circle is often referred to as the $\boxed{\text{unit circle}}$.
Alternatively,

$$\begin{aligned} |z| = 1 \quad &\Rightarrow \quad \sqrt{x^2 + y^2} = 1 \\ &\Rightarrow \quad x^2 + y^2 = 1 \end{aligned}$$

which is the equation of a circle, radius 1 and centre at the origin.
If $|z|$ satisfies

$$|z - z_0| = r$$

(where z_0 is a fixed complex number and r is a fixed positive number) then z lies on the circle of radius r with centre at z_0.
To see this, let

$$z = x + iy \quad \text{and} \quad z_0 = x_0 + iy_0$$

then

$$z - z_0 = (x - x_0) + i(y - y_0)$$

and so

$$\begin{aligned} |z - z_0| = r \quad &\Rightarrow \quad \sqrt{(x - x_0)^2 + (y - y_0)^2} = r \\ &\Rightarrow \quad (x - x_0)^2 + (y - y_0)^2 = r^2 \end{aligned}$$

which is the equation of a circle, centre (x_0, y_0) and radius r.

Example: If z satisfies the equation

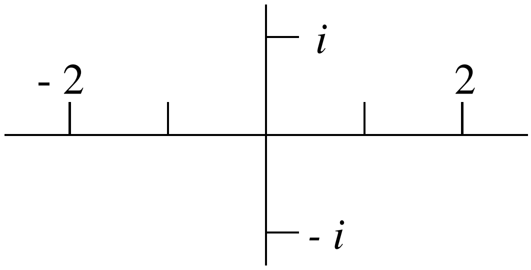
$$|z - i| = |z + i| \qquad (\star)$$

what is the locus of z in the Argand diagram?

Solution 1: Geometry

The given equation implies that the distance between the point P representing z and the point i is the same as that between P and $-i$. It follows that P lies on the perpendicular bisector of the points i and $-i$.

So P lies on the real axis.



Solution 2: Analysis

Set $z = x + iy$ so that the equation $(\star)$ becomes

$$\begin{aligned}
|x + i(y-1)| &= |x + i(y+1)| \\
\Rightarrow \sqrt{x^2 + (y-1)^2} &= \sqrt{x^2 + (y+1)^2} \\
\Rightarrow x^2 + (y-1)^2 &= x^2 + (y+1)^2 \\
\Rightarrow x^2 + y^2 - 2y + 1 &= x^2 + y^2 + 2y + 1 \\
\Rightarrow 4y &= 0 \\
\Rightarrow y &= 0
\end{aligned}$$

Thus the imaginary part of z is zero and so z is a real number and hence lies on the real axis.

Example: If $|z + 1| = \sqrt{2}|z - 1|$, what is the locus of z?

Solution

Let $z = x + iy$, then

$$\begin{aligned}
|z+1| &= |(x+1) + iy| = \sqrt{(x+1)^2 + y^2} \\
\text{and} \quad |z-1| &= |(x-1) + iy| = \sqrt{(x-1)^2 + y^2}
\end{aligned}$$

Thus

$$
\begin{aligned}
(x+1)^2 + y^2 &= 2[(x-1)^2 + y^2] \\
x^2 + 2x + 1 + y^2 &= 2[x^2 - 2x + 1 + y^2] \\
&= 2x^2 - 4x + 2 + 2y^2 \\
x^2 + y^2 - 6x + 1 &= 0 \\
(x-3)^2 - 9 + y^2 + 1 &= 0 \\
\Rightarrow (x-3)^2 + y^2 &= 8
\end{aligned}
$$

Thus the point representing z in the Argand diagram lies on a circle with radius $\sqrt{8}$ and centred on $(3, 0)$. This result is known as the circle of Apollonius.

Example

Show that multiplying any complex number by i has the same effect as an anticlockwise rotation of $\pi/2$.

Solution

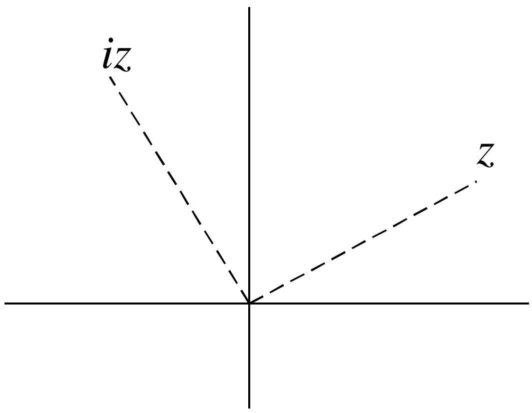
If z has modulus r and argumentθ then

$$z = \langle r, \theta \rangle \text{ and } i = \langle 1, \pi/2 \rangle$$

and so

$$
\begin{aligned}
iz &= \langle r, \theta \rangle \langle 1, \pi/2 \rangle \\
&= \langle r \times 1, \theta + \pi/2 \rangle \\
&= \langle r, \theta + \pi/2 \rangle
\end{aligned}
$$

So iz has the same modulus as z but the argument is increased by $\pi/2$.



1.5 De Moivre's Theorem

Let z_1 have modulus r_1 and argument θ_1. Also let z_2 have modulus r_2 and argument θ_2. Then

$$\begin{aligned} z_1 z_2 &= \langle r_1, \theta_1 \rangle \langle r_2, \theta_2 \rangle \\ &= \langle r_1 r_2, \theta_1 + \theta_2 \rangle \end{aligned}$$

If $\theta_1 = \theta_2 = \theta$ and $r_1 = r_2 = r$ then we see that

$$\langle r, \theta \rangle^2 = \langle r^2, 2\theta \rangle$$

and so

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i \sin 2\theta$$

Also

$$\begin{aligned} \langle r, \theta \rangle^3 &= \langle r^2, 2\theta \rangle \langle r, \theta \rangle \\ &= \langle r^3, 3\theta \rangle \end{aligned}$$

and

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$$

for any natural number n.
The generalization of this result is de Moivre's theorem which states that

$$(\cos\theta + i\sin\theta)^q = \cos q\theta + i \sin q\theta$$

for any rational number q.
More accurately, the r.h.s. gives one of the values of the l.h.s. If $q = m/n$ (where m and n are integers, n positive) then the l.h.s. has n possible values whereas the r.h.s. has just one value.

1.6 Euler's Relation

If x is any real number, then

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \end{aligned}$$

These expansions are now taken to apply to all complex numbers. Thus

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \dots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \dots\right) \\ &= \cos\theta + i\sin\theta \end{aligned}$$

which gives Euler's relation,

$$\boxed{\boxed{e^{i\theta} = \cos\theta + i\sin\theta}}$$

Note

(a) $e^{-i\theta} = \cos\theta - i\sin\theta$,

(b) $e^{i\pi/2} = i$,

(c) $e^{i\pi} = -1$,

(d) $e^{i3\pi/2} = -i$,

(e) $e^{i2p\pi} = 1$ for any integer p.

From the results

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \text{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

it follows that

$$\boxed{\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)}$$

and

$$\boxed{\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)}$$

Also, de Moivre's theorem can now be written as

$$\boxed{\left(e^{i\theta}\right)^q = e^{iq\theta}}$$

for any rational number q.

Example: Find the real and imaginary parts of

$$z = \frac{(\sqrt{3} + i)^6}{(i - 1)^3}.$$

Solution

Let $z_1 = (\sqrt{3} + i)$ and $z_2 = (i - 1)$, and write z_1 and z_2 in polar form:

$$|z_1| = |\sqrt{3} + i| = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2$$

and if Arg$z_1 = \theta$ then

$$\cos\theta = \frac{\sqrt{3}}{2}, \ \ \sin\theta = \frac{1}{2}, \ \ -\pi < \theta \leq \pi \Rightarrow \theta = \frac{\pi}{6}$$

Thus $z_1 = (\sqrt{3} + i) = 2e^{i\pi/6}$.
Also

$$|z_2| = |i - 1| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

and if Arg$z_2 = \phi$ then

$$\cos\phi = -\frac{1}{\sqrt{2}}, \ \ \sin\phi = \frac{1}{\sqrt{2}}, \ \ -\pi < \phi \leq \pi \Rightarrow \phi = \frac{3\pi}{4}$$

Thus $z_2 = (i - 1) = \sqrt{2}e^{i3\pi/4}$.
The numerator of z is therefore

$$(\sqrt{3} + i)^6 = \left(2e^{i\pi/6}\right)^6 = 2^6 e^{6i\pi/6} = 2^6 e^{-i\pi} = -2^6$$

Also, the denominator of z is

$$\begin{aligned}(i-1)^{-3} &= \left(\sqrt{2}e^{i3\pi/4}\right)^{-3} = 2^{-3/2}e^{-9i\pi/4} = 2^{-3/2}e^{-i\pi/4} \\ &= 2^{-3/2}\left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right] = 2^{-3/2}\left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right] = 2^{-2}(1-i)\end{aligned}$$

Therefore

$$z = \frac{(\sqrt{3} + i)^6}{(i-1)^3} = -2^6 \times 2^{-2}(1-i) = 16(i-1) \quad \Longrightarrow \quad Re(z) = -16, Im(z) = 16.$$

1.7 Applications of De Moivre's Theorem

Example: Show that each of

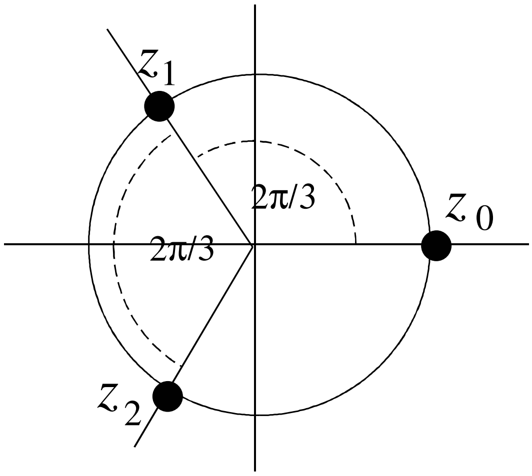
$$z_0 = \langle 1, 0\rangle, \ \ z_1 = \langle 1, 2\pi/3\rangle, \ \ z_2 = \langle 1, 4\pi/3\rangle$$

satisfies the equation $z^3 = 1$. Plot all 3 numbers in the Argand diagram.

Solution
Firstly $z_0 = 1 \Rightarrow z_0^3 = 1$.
Also, since $z = \langle r, \theta\rangle = r(\cos\theta + i\sin\theta)$
$\Rightarrow z^3 = \langle r^3, 3\theta\rangle = r^3(\cos 3\theta + i\sin 3\theta)$
it follows that

$$\begin{aligned} z_1^3 = \langle 1^3, 3\times 2\pi/3\rangle &= \langle 1, 2\pi\rangle = 1 \\ \text{and} \ \ z_2^3 = \langle 1^3, 3\times 4\pi/3\rangle &= \langle 1, 4\pi\rangle = 1\end{aligned}$$

Since all of the z's have modulus 1, they lie on the unit circle. They are called the $\boxed{\text{cube roots}}$ of unity.



In the last example we verified that three $\boxed{\text{given}}$ numbers were the cube roots of unity. We now show how to find them directly. The method used will allow us to find $\boxed{\text{all}}$ the roots of any complex number.

Example: Find the cube roots of unity, i.e. the three values of $1^{1/3}$.

Solution

First write 1 in the $\boxed{\text{most general}}$ polar form. Since $|1| = 1$ and Arg$(1) = 0$ it follows that $\arg(1) = 2p\pi$ for any $\boxed{\text{integer}}$ p and that

$1 = 1[\cos(2p\pi) + i\sin(2p\pi)]$ is the most general polar form of the (complex) number 1. Now we require $z = 1^{1/3}$ and so, by de Moivre's theorem,

$$\begin{aligned} z &= \{1[\cos(2p\pi) + i\sin(2p\pi)]\}^{1/3} \\ &= \cos\left(\frac{2p\pi}{3}\right) + i\sin\left(\frac{2p\pi}{3}\right) \end{aligned}$$

for any integer p. Setting p equal to 0, 1 and 2 in turn gives the required 3 values. It is usual to denote the corresponding z values by the subscript p.

$$\begin{aligned} p = 0 \quad &\Rightarrow \quad z_0 = \cos 0 + i\sin 0 = 1 \\ p = 1 \quad &\Rightarrow \quad z_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ p = 2 \quad &\Rightarrow \quad z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}$$

These are the 3 numbers in the previous example.

Note

(*a*) Any other value of p will give one of $z_0,\ z_1,\ z_2$. Thus there are exactly 3 distinct values of z for which $z^3 = 1$.

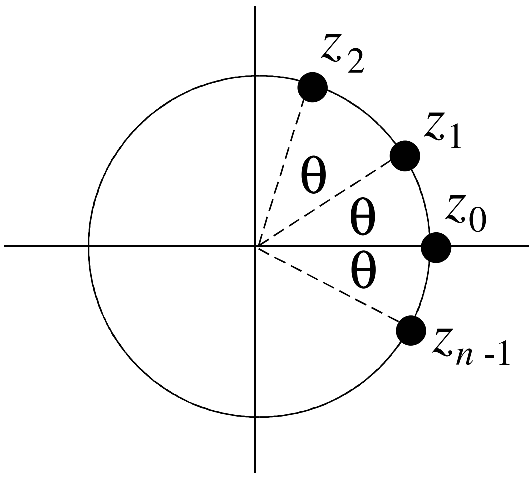
(*b*) z_1 and z_2 form a complex conjugate pair. This illustrates a general result, viz. that if z is a root of a polynomial with $\boxed{\text{real}}$ coefficients, the $\bar{z}$ is also a root. (Of course, if z is real then this says nothing.)

1.7.1 The nth Roots of Unity

If n is any positive integer, we say that z is an nth root of unity if $z^n = 1$. To find z, proceed as before:

$$\begin{aligned} 1 &= 1[\cos(2p\pi) + i\sin(2p\pi)] \quad (p \text{ any integer}) \\ z &= 1^{1/n} = [\cos(2p\pi) + i\sin(2p\pi)]^{1/n} \\ &= \cos\left(\frac{2p\pi}{n}\right) + i\sin\left(\frac{2p\pi}{n}\right) \end{aligned}$$

Taking $p = 0, 1, 2, \ldots, (n-1)$ will give the n roots. (Other values of n will just repeat the values.) They all have modulus 1 and so lie on the unit circle.



The angle θ is $2\pi/n$. The roots are equally spaced around the unit circle.

Example: Find all values z for which

$$z^3 = 1 - i\sqrt{3}$$

Solution: First write $1 - i\sqrt{3}$ in the most general polar form. Since

$$|1 - i\sqrt{3}| = \sqrt{1^2 + (-\sqrt{3})^2} = \sqrt{4} = 2$$

and, for the principal argument θ,

$$\cos\theta = \frac{1}{2}, \ \sin\theta = -\frac{\sqrt{3}}{2}, \ -\pi < \theta \leq \pi \Rightarrow \theta = -\frac{\pi}{3}$$

Thus

$$z^3 = 2\left[\cos\left(-\frac{\pi}{3} + 2p\pi\right) + i\sin\left(-\frac{\pi}{3} + 2p\pi\right)\right]$$

where p is any integer. De Moivre's theorem now gives

$$z = 2^{1/3}\left[\cos\left(-\frac{\pi}{9} + \frac{2p\pi}{3}\right) + i\sin\left(-\frac{\pi}{9} + \frac{2p\pi}{3}\right)\right]$$

Now set p equal to 0,1,2 in turn to give the three values for z:

$$\begin{aligned}
z_0 &= 2^{1/3}\left[\cos\left(-\frac{\pi}{9}\right) + i\sin\left(-\frac{\pi}{9}\right)\right] \\
z_1 &= 2^{1/3}\left[\cos\left(\frac{5\pi}{9}\right) + i\sin\left(\frac{5\pi}{9}\right)\right] \\
z_2 &= 2^{1/3}\left[\cos\left(\frac{11\pi}{9}\right) + i\sin\left(\frac{11\pi}{9}\right)\right]
\end{aligned}$$

$z_0,\ z_1,\ z_2$ are equally spaced around the circle $|z| = 2^{1/3}$.

1.7.2 Deriving multi-angle trigonometric formulae using de Moivre's theorem

De Moivre's theorem (and the Binomial expansion) can be used to easily find expressions for multi-angle trigonometric functions. For any positive integer n, de Moivre's theorem states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Equating real and imaginary parts, we see that

$$\cos n\theta = Re\left[(\cos\theta + i\sin\theta)^n\right], \qquad \sin n\theta = Im\left[(\cos\theta + i\sin\theta)^n\right].$$

Example: Expand $\cos 2\theta$ and $\sin 2\theta$ as powers of $\cos\theta$ and $\sin\theta$.

Solution: Since $\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2$, we see that $\cos 2\theta$ and $\sin 2\theta$ are the real and imaginary parts of $(\cos\theta + i\sin\theta)^2$. Thus we expand $(\cos\theta + i\sin\theta)^2$ by the binomial theorem and equate real and imaginary parts:

$$\begin{aligned}
(\cos\theta + i\sin\theta)^2 &= \cos^2\theta + 2i\cos\theta\sin\theta + (i\sin\theta)^2 \\
&= \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta.
\end{aligned}$$

Hence

$$\begin{aligned}
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
\sin 2\theta &= 2\sin\theta\cos\theta.
\end{aligned}$$

Using $\cos^2\theta + \sin^2\theta = 1$, we can also write $\cos 2\theta$ as

$$\begin{aligned}
\cos 2\theta &= 2\cos^2\theta - 1 \\
\text{or}\quad \cos 2\theta &= 1 - 2\sin^2\theta.
\end{aligned}$$

Example: Expand $\cos 3\theta$ as a sum of powers of $\cos\theta$.

Solution: From de Moivre's theorem with $n = 3$ we have

$$\cos 3\theta + i\sin 3\theta = (\cos\theta + i\sin\theta)^3$$

and so $\cos 3\theta$ is the real part of $(\cos\theta + i\sin\theta)^3$. Again we use the Binomial Theorem to expand this expression:

$$\begin{aligned}(\cos\theta + i\sin\theta)^3 &= (\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta + 3i\sin\theta\cos^2\theta - 3\sin^2\theta\cos\theta - i\sin^3\theta.\end{aligned}$$

Hence $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$. Since $\cos^2\theta + \sin^2\theta = 1$, we can write $\cos 3\theta$ as

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= 4\cos^3\theta - 3\cos\theta.\end{aligned}$$

Exercise: Show that

$$\begin{aligned}\sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\ \cos 4\theta &= 8\cos^4\theta - 8\cos^2\theta + 1 \\ \sin 4\theta &= 4\cos\theta(\sin\theta - 2\sin^3\theta).\end{aligned}$$

1.7.3 Finding the m^{th} roots of a complex number

De Moivre's theorem holds when n is rational (and not just when n is an integer). However, in this case, $\cos n\theta + i\sin n\theta$ is just one of several values that $(\cos\theta + i\sin\theta)^n$ may have. Suppose that $n = 1/m$ (where m is an integer), then

$$(\cos\theta + i\sin\theta)^{1/m} = \cos\frac{\theta}{m} + i\sin\frac{\theta}{m}.$$

This gives one of the m^{th} roots of the complex number $z = \cos\theta + i\sin\theta$. However, there are a total of m distinct m^{th} roots of z. We can obtain them all by writing z in the form

$$z = \cos\theta + i\sin\theta = \cos(\theta + 2\pi p) + i\sin(\theta + 2\pi p)$$

where p is any integer. De Moivre's theorem now gives

$$\begin{aligned}z^{1/m} &= [\cos(\theta + 2\pi p) + i\sin(\theta + 2\pi p)]^{1/m} \\ &= \cos\left(\frac{\theta + 2\pi p}{m}\right) + i\sin\left(\frac{\theta + 2\pi p}{m}\right).\end{aligned}$$

Taking $p = 0,\ 1,\ 2, \ldots, m-1$ gives all the different roots. (Other values of p could be used, but they will just repeat previous roots.) We can extend this method to any complex number by writing it in the form

$$z = r(\cos\theta + i\sin\theta),\ \ r > 0$$

(modulus-argument form) or

$$z = r\{\cos(\theta + 2\pi p) + i\sin(\theta + 2\pi p)\},\ \ r > 0$$

where again p is any integer. Applying de Moivre's theorem now gives

$$\begin{aligned}z^{1/m} &= r^{1/m}\,[\cos(\theta + 2\pi p) + i\sin(\theta + 2\pi p)]^{1/m} \\ &= r^{1/m}\left[\cos\left(\frac{\theta + 2\pi p}{m}\right) + i\sin\left(\frac{\theta + 2\pi p}{m}\right)\right].\end{aligned}$$

Alternatively we may use the exponential notation and then

$$z = re^{i\theta} = re^{i(\theta+2\pi p)},$$

so that

$$\begin{aligned} z^{1/m} &= r^{1/m} \exp\left[i\left(\frac{\theta+2\pi p}{m}\right)\right] \\ &= r^{1/m}\left[\cos\left(\frac{\theta+2\pi p}{m}\right) + i\sin\left(\frac{\theta+2\pi p}{m}\right)\right], \end{aligned}$$

where $p = 0,\ 1,\ 2, \ldots, m-1$.

Example: Find the two square roots of $z = 4i$.

Solution: In modulus-argument (polar) form,

$$z = 4\left[\cos\left(\frac{\pi}{2} + 2\pi p\right) + i\sin\left(\frac{\pi}{2} + 2\pi p\right)\right]$$

for any integer p. Therefore, using de Moivre's theorem,

$$z^{1/2} = 4^{1/2}\left[\cos\left(\frac{\pi}{4} + \pi p\right) + i\sin\left(\frac{\pi}{4} + \pi p\right)\right],\ p = 0, 1.$$

$$p = 0 \ \Rightarrow\ z^{1/2} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2} + i\sqrt{2}$$

$$p = 1 \ \Rightarrow\ z^{1/2} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 2\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = -\sqrt{2} - i\sqrt{2}$$

Alternatively, we may use the exponential notation:

$$z = 4e^{i(\pi/2+2\pi p)} \ \Rightarrow\ z^{1/2} = 4^{1/2}e^{i(\pi/4+\pi p)} = 2e^{i(\pi/4+\pi p)},\ p = 0, 1$$

and now substitute in the values for p to get

$$p = 0 \ \Rightarrow\ z^{1/2} = 2e^{i\pi/4} = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2}$$

$$p = 1 \ \Rightarrow\ z^{1/2} = 2e^{i5\pi/4} = 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$$

Example: Find the cube roots of $z = \sqrt{3} + i$ and show them on an Argand diagram.

Solution: In polar form

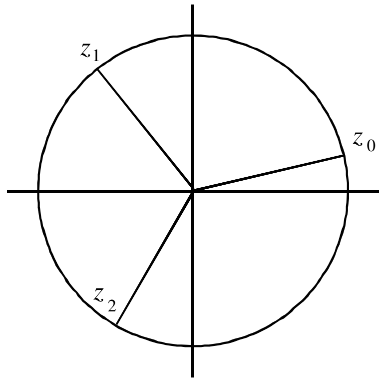
$$\begin{aligned} z &= 2\left[\cos\left(\frac{\pi}{6} + 2\pi p\right) + i\sin\left(\frac{\pi}{6} + 2\pi p\right)\right] \\ \Rightarrow z^{1/3} &= 2^{1/3}\left[\cos\left(\frac{\pi}{18} + \frac{2\pi p}{3}\right) + i\sin\left(\frac{\pi}{18} + \frac{2\pi p}{3}\right)\right],\ p \ = \ 0, 1, 2 \end{aligned}$$

$$p = 0 \ \Rightarrow\ z_0 = z^{1/3} = 2^{1/3}\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)$$

$$p = 1 \ \Rightarrow\ z_1 = z^{1/3} = 2^{1/3}\left(\cos\frac{13\pi}{18} + i\sin\frac{13\pi}{18}\right)$$

$$p = 2 \ \Rightarrow\ z_2 = z^{1/3} = 2^{1/3}\left(\cos\frac{25\pi}{18} + i\sin\frac{25\pi}{18}\right)$$

Argand Diagram to show the roots on the circle $|z| = 2^{1/3}$.



Example: Find the four distinct fourth roots of $z = -1$.

Solution: In exponential form,

$$\begin{aligned} z &= e^{i(\pi+2\pi p)} \\ \Rightarrow\ z^{1/4} &= e^{i(2p+1)\pi/4} \end{aligned}$$

for any integer p. Therefore

$$p = 0 \quad \Rightarrow \quad z_0 = z^{1/4} = e^{i\pi/4} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$$

$$p = 1 \quad \Rightarrow \quad z_1 = z^{1/4} = e^{i3\pi/4} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{-1+i}{\sqrt{2}}$$

$$p = 2 \quad \Rightarrow \quad z_2 = z^{1/4} = e^{i5\pi/4} = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = \frac{-1-i}{\sqrt{2}}$$

$$p = 3 \quad \Rightarrow \quad z_3 = z^{1/4} = e^{i7\pi/4} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{1-i}{\sqrt{2}}$$

To summarise, there are three steps to carry out to find the m^{th} roots of a complex number:

1. Write the complex number z in general polar form

$$\begin{aligned} z &= r[\cos(\theta + 2\pi p) + i\sin(\theta + 2\pi p)],\ r > 0 \\ \text{or}\ \ z &= re^{i(\theta+2\pi p)} \end{aligned}$$

2. Use de Moivre's theorem

$$\begin{aligned} z^{1/m} &= r^{1/m}\left[\cos\left(\frac{\theta + 2\pi p}{m}\right) + i\sin\left(\frac{\theta + 2\pi p}{m}\right)\right] \\ \text{or}\ \ z^{1/m} &= r^{1/m}e^{i(\theta+2\pi p)/m} \end{aligned}$$

3. Let $p = 0,\ 1,\ 2, \ldots, m-1$ to get the m roots.

1.7.4 Expressing Powers of Trigonometric Functions as Multiple Angle Functions

It is sometimes useful to be able to express $\cos^n\theta$ and $\sin^n\theta$, where n is a positive integer, in terms of cosines and sines of multiples of θ. This is necessary when such expressions have to be integrated. To achieve this we use the Binomial expansion and the identities

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \ \ \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

.

Example: Express $\cos^2\theta$ and $\sin^2\theta$ in terms of $\cos 2\theta$.

Solution:

$$\begin{aligned}\cos^2\theta = (\cos\theta)^2 &= \left[\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right]^2 \\ &= \frac{1}{4}\left[(e^{i\theta})^2 + 2e^{i\theta}.e^{-i\theta} + (e^{-i\theta})^2\right] \\ &= \frac{1}{4}(e^{2i\theta} + 2 + e^{-2i\theta}).\end{aligned}$$

But $e^{2i\theta} + e^{-2i\theta} = 2\cos 2\theta$, and so

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta).$$

We can then use the relation $\sin^2\theta = 1 - \cos^2\theta$ to deduce that

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta).$$

Example: Express $\cos^3\theta$ in multiple angles.

Solution:

$$\begin{aligned}\cos^3\theta &= (\cos\theta)^3 = \left[\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right]^3 \\ &= \frac{1}{8}(e^{3i\theta} + 3e^{2i\theta}.e^{-i\theta} + 3e^{i\theta}.e^{-2i\theta} + e^{-3i\theta}) \\ &= \frac{1}{8}(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}) \\ &= \frac{1}{8}\left[(e^{3i\theta} + e^{-3i\theta}) + 3(e^{i\theta} + e^{-i\theta})\right] \\ &= \frac{1}{8}\left[2\cos 3\theta + 6\cos\theta\right] \\ &= \frac{1}{4}(\cos 3\theta + 3\cos\theta).\end{aligned}$$

Exercise: Find

1. the fifth roots of $-6\sqrt{3}+6i$
2. the sixth roots of -1
3. the fifth roots of i
4. the cube roots of $1+i$.

Except for (2), leave the answer in polar form.

Answers

1. $1{\cdot}64(\cos\phi + i\sin\phi)$, where $\phi = \frac{\pi}{6}, \frac{17\pi}{30}, \frac{29\pi}{30}, \frac{41\pi}{30}, \frac{53\pi}{30}$.
2. $\pm i, \frac{\sqrt{3}}{2} \pm \frac{i}{2}, -\frac{\sqrt{3}}{2} \pm \frac{i}{2}$.
3. $\cos\phi + i\sin\phi$, where $\phi = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$.
4. $1{\cdot}12(\cos\phi + i\sin\phi)$, where $\phi = \frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}$.

Exercise: Show that

1. $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$
2. $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$
3. $\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3)$

In all of these examples, the expansion of a power of cosine gives a series involving only *cosines* of multiples of the angle. This is also true for an *even* power of sine, but an *odd* power of sine gives a series with *sines* of multiple angles. This reflects the fact that both $\cos^n\theta$ and $\sin^{2n}\theta$ are *even functions* (i.e. their value is unchanged if θ is replaced by $-\theta$) but $\sin^{2n+1}\theta$ is an *odd function* (its value changes sign if θ is replaced by $-\theta$).

1.8 Circular and Hyperbolic Functions of Complex Numbers

We have already shown that

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Also the definitions of the basic hyperbolic functions are

$$\begin{aligned}\cosh x &= \frac{1}{2}\left(e^x + e^{-x}\right)\\ \sinh x &= \frac{1}{2}\left(e^x - e^{-x}\right)\end{aligned}$$

If these last two definitions are also taken to apply to complex numbers, then we have

$$\begin{aligned}\cosh i\theta &= \cos\theta\\ \sinh i\theta &= i\sin\theta\end{aligned}$$

and also

$$\begin{aligned}\cos i\theta &= \cosh\theta\\ \sin i\theta &= i\sinh\theta\end{aligned}$$

The results are true for any complex number θ (not just real and pure-imaginary). All of the standard trigonometric identities are valid for all complex numbers. For example, from

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

it follows that

$$\begin{aligned}\sin(x+iy) &= \sin x\cos(iy) + \cos x\sin(iy)\\ &= \sin x\cosh y + i\cos x\sinh y\end{aligned}$$

So, if x and y are real, then

$$\begin{aligned}Re(\sin(x+iy)) &= \sin x\cosh y\\ Im(\sin(x+iy)) &= \cos x\sinh y\end{aligned}$$

Example: Find a z for which $\sin z = 2$.

Solution: Let $z = x + iy$, then we require

$$\begin{aligned}Re(\sin(x+iy)) &= \sin x\cosh y = 2\\ Im(\sin(x+iy)) &= \cos x\sinh y = 0\end{aligned}$$

and these are satisfied by $x = \pi/2 = 1.57079$ and $y = \cosh^{-1} 2 = 1.31695$.