University of Sheffield

School of Mathematics and Statistics

MAS140: Mathematics (Chemical)
MAS152: Civil Engineering Mathematics
MAS152: Essential Mathematical Skills & Techniques
MAS156: Mathematics (Electrical and Aerospace)
MAS161: General Engineering Mathematics

Semester 1 2016–17

Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,
`http://engmaths.group.shef.ac.uk/mas140/`
`http://engmaths.group.shef.ac.uk/mas151/`
`http://engmaths.group.shef.ac.uk/mas152/`
`http://engmaths.group.shef.ac.uk/mas156/`
`http://engmaths.group.shef.ac.uk/mas161/`
which can also be accessed through MOLE.

1 Partial Differentiation

We can extend the idea of differentiation to include functions of two or more **independent** variables. For example, if the value of z depends upon both of the values x and y (and y is **not** a function of x) then z is said to be a function of x and y and we write

$$z = f(x, y).$$

A specific example of a function of two independent variables is given by the volume $V(h, r)$ of a cylinder of height h and cross-sectional radius r:

$$V(h, r) = \pi r^2 h,$$

while the volume of a cuboidal box with sides x, y and z is a function of three variables:

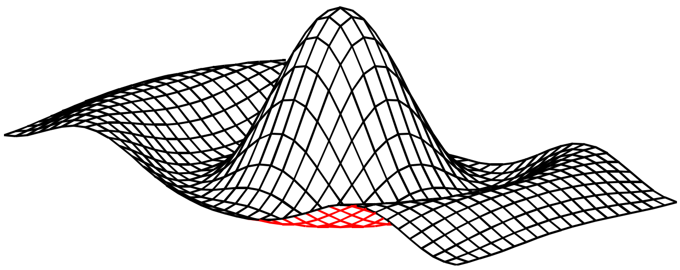
$$V(x, y, z) = xyz.$$

Graphical representation of functions of two variables

Just as a function of one independent variable can be thought of as a curve, a function of **two** independent variables can be thought of as a 3-dimensional surface, with the value of the function giving the height of the surface above a 2-dimensional plane specified by the independent variables. For example, the function

$$z(x, y) = \frac{\sin\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$$

is shown below.



Definition

Previously, we considered the derivative of a function of a single variable as the sensitivity of the dependent variable (the output) to a change in the independent variable (the input). For a function of several independent variables, we can consider the sensitivity of the dependent variable to a change in just one of the independent variables, **while the other independent variables are held constant**.

So, for the example of the volume of a cylinder, we can ask how sensitive the volume is to a change in the height (with the radius held constant), **or** to changes in the radius (with the height held constant).

We formalise this by introducing the notion of **partial derivatives**. We define this for a function of two independent variables, but the extension of the definition to functions of more than two independent variables is straightforward.
The **partial derivative** of $f(x, y)$ with respect to (w.r.t.) x is defined by

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

(provided the limit exists). Note that y is **held constant**, while x varies. Similarly, the partial derivative of f w.r.t. y is

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

where now x is kept constant.

We find partial derivatives in the same way as find derivatives of functions of a single variable, since we are differentiating with respect to only one variable at a time, the other variables being held constant. Furthermore, we can use the product and quotient rules as before.

Example: Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when

(i) $f(x, y) = x^3 + y^2$,
(ii) $f(x, y) = x \cos y$,
(iii) $f(x, y) = x \tan^{-1} y + x^3 y + y^{3/2}$.

Solution:

(i) $\dfrac{\partial f}{\partial x} = \dfrac{\partial x^3}{\partial x} + \dfrac{\partial y^2}{\partial x} = 3x^2 + 0 = 3x^2$ (y is held constant when differentiating w.r.t. x.)

$\dfrac{\partial f}{\partial y} = \dfrac{\partial x^3}{\partial y} + \dfrac{\partial y^2}{\partial y} = 0 + 2y = 2y$ (x is held constant when differentiating w.r.t. y.)

(ii) $\dfrac{\partial f}{\partial x} = \dfrac{\partial x}{\partial x} \cos y + x \dfrac{\partial \cos y}{\partial x} = \cos y + 0 = \cos y$ (using the Product Rule)

$\dfrac{\partial f}{\partial y} = \dfrac{\partial x}{\partial y} \cos y + x \dfrac{\partial \cos y}{\partial y} = 0 - x \sin y = -x \sin y$ (using the Product Rule)

(iii)
$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x \tan^{-1} y\right) + \frac{\partial}{\partial x}\left(x^3 y\right) + \frac{\partial}{\partial x}\left(y^{3/2}\right) \\ &= \tan^{-1} y + 3x^2 y \qquad \text{(using the Product Rule)} \end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x \tan^{-1} y\right) + \frac{\partial}{\partial y}\left(x^3 y\right) + \frac{\partial}{\partial y}\left(y^{3/2}\right) \\ &= \frac{x}{1+y^2} + x^3 + \frac{3}{2} y^{1/2} \qquad \text{(using the Product Rule)} \end{aligned}$$

Example: If $f(x,y) = x\cos(xy)$, find $\dfrac{\partial f}{\partial x}$.

Solution: $\dfrac{\partial f}{\partial x} = x\dfrac{\partial}{\partial x}[\cos(xy)] + \cos(xy)\dfrac{\partial x}{\partial x}$ (product rule)

$$\begin{aligned}
\frac{\partial \cos(xy)}{\partial x} &= \frac{\partial \cos u}{\partial x} \quad \text{(where } u = xy\text{)} \\
&= \frac{d\cos u}{du}\frac{\partial u}{\partial x} \quad \text{(chain rule)} \\
&= -\sin u \frac{\partial (xy)}{\partial x} = -y\sin(xy)
\end{aligned}$$

and so

$$\frac{\partial f}{\partial x} = -xy\sin(xy) + \cos(xy).$$

1.1 Higher Order Derivatives

Since $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ are (usually) functions of both x and y, each may be partially differentiated in two different ways:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right).$$

These **second order partial derivatives** are usually written

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y^2} \qquad \text{or} \qquad f_{xx}, \quad f_{xy}, \quad f_{yx}, \quad f_{yy}$$

respectively. Note the orders of the x and y in the two notations.

Example: Find the 4 second order partial derivatives of
(i) $f(x,y) = x^3 + y^2$
(ii) $f(x,y) = x\tan^{-1} y + x^3 y + y^{3/2}$

Solution:

(i) $\dfrac{\partial f}{\partial x} = 3x^2$ and $\dfrac{\partial f}{\partial y} = 2y$ and so

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial(3x^2)}{\partial x} = 6x \\
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial(3x^2)}{\partial y} = 0 \\
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial(2y)}{\partial y} = 2 \\
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial(2y)}{\partial x} = 0
\end{aligned}$$

(ii) $f(x,y) = x\tan^{-1} y + x^3 y + y^{3/2}$ and so

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}(\tan^{-1} y + 3x^2 y) = 6xy \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}(\tan^{-1} y + 3x^2 y) = \frac{1}{1+y^2} + 3x^2 \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{x}{1+y^2} + x^3 + \frac{3}{2}y^{1/2}\right) \\
&= \frac{1}{1+y^2} + 3x^2 \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{x}{1+y^2} + x^3 + \frac{3}{2}y^{1/2}\right) \\
&= -\frac{2xy}{(1+y^2)^2} + \frac{3}{4}y^{-1/2}
\end{aligned}$$

Note: In these examples $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

This result is true for all of the functions that we shall meet and it may be assumed **except** when you are explicitly asked to verify that it is true (which often occurs!).

Second partial derivatives often occur in models of physical and chemical processes. For example, the differential equation satisfied by the concentration $C(x,t)$ of a substance diffusing in one spatial dimension (x) is

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}, \quad D = \text{constant},$$

where t is time.

Example: If $f(x,y) = x^2 \tan^{-1}(y/x)$, find $\dfrac{\partial^2 f}{\partial x \partial y}$ at $(1,1)$.

Solution:

$$\begin{aligned}
\frac{\partial f}{\partial y} &= x^2 \frac{\partial}{\partial y} \tan^{-1} u \quad (\text{where} \quad u = y/x) \\
&= x^2 \frac{d}{du}(\tan^{-1} u) \frac{\partial u}{\partial y} = \frac{x^2}{(1+u^2)} \frac{\partial}{\partial y}\left(\frac{y}{x}\right) \\
&= \frac{x^2}{\left[1 + \left(\frac{y}{x}\right)^2\right]} \frac{1}{x} = x^2 \frac{x^2}{(x^2+y^2)} \frac{1}{x} = \frac{x^3}{x^2+y^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{x^3}{x^2+y^2}\right) \\
&= \frac{3x^2}{x^2+y^2} + x^3(-1)\frac{1}{(x^2+y^2)^2} 2x \\
&= \frac{3x^2}{x^2+y^2} - \frac{2x^4}{(x^2+y^2)^2} = \frac{3x^4 + 3y^2x^2 - 2x^4}{(x^2+y^2)^2}
\end{aligned}$$

and so

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^4 + 3x^2y^2}{(x^2+y^2)^2}.$$

Hence, the value of $\dfrac{\partial^2 f}{\partial x \partial y}$ at $(1,1)$ is $\dfrac{1+3}{(1+1)^2} = 1$.