

MAS140: Mathematics (Chemical)

MAS152: Civil Engineering Mathematics

MAS152: Essential Mathematical Skills & Techniques

MAS156: Mathematics (Electrical and Aerospace)

MAS161: General Engineering Mathematics

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## Semester 1 2017–18

### Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

### Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,

<http://engmaths.group.shef.ac.uk/mas140/>

<http://engmaths.group.shef.ac.uk/mas151/>

<http://engmaths.group.shef.ac.uk/mas152/>

<http://engmaths.group.shef.ac.uk/mas156/>

<http://engmaths.group.shef.ac.uk/mas161/>

which can also be accessed through MOLE.

# 1 Maclaurin and Taylor Series

## 1.1 Maclaurin Series

We say that a function  $f(x)$  possesses a **power series expansion** if there exist numbers  $a_0, a_1$  etc., which are independent of  $x$ , such that

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots \quad (1.1)$$

Suppose that  $f(x)$  does possess a power series expansion. Then setting  $x = 0$  gives  $a_0 = f(0)$ . If we further assume that the power series expansion can be differentiated, we have

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \dots,$$

and setting  $x = 0$  in this expression gives  $a_1 = f'(0)$ .

Repeating the process (of differentiating and then setting  $x = 0$ ), we have

$$f''(x) = 2a_2 + 3 \cdot 2a_3x + \dots \quad \Rightarrow \quad a_2 = \frac{f''(0)}{2}$$

$$f'''(x) = 3 \cdot 2a_3 + 4 \cdot 3 \cdot 2a_4x + \dots \quad \Rightarrow \quad a_3 = \frac{f'''(0)}{3 \cdot 2}$$

$$f^{(4)}(x) = 4 \cdot 3 \cdot 2a_4 + \dots \quad \Rightarrow \quad a_4 = \frac{f^{(4)}(0)}{4 \cdot 3 \cdot 2}.$$

The general coefficient in the power series is

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{for } n = 0, 1, 2, 3, \dots$$

(Recall that  $0! = 1$  and  $f^{(n)}$  is the  $n$ th derivative of  $f$ .)

Having obtained an expression for all the  $a_n$ , we can substitute these into Eq. (1.1) to obtain

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

This is the **Maclaurin series** (or expansion) of  $f$ .

### Examples

1.  $f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x \quad \Rightarrow \quad f^{(n)}(x) = e^x$

Thus  $f^{(n)}(0) = 1$  for  $n = 0, 1, 2, \dots$ , and so

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

2.  $f(x) = \sin x$

$$f'(x) = \cos x, f^{(2)}(x) = -\sin x, f^{(3)}(x) = -\cos x, f^{(4)}(x) = \sin x, \dots$$

Therefore  $f(0) = 0, f^{(1)}(0) = 1, f^{(2)}(0) = 0, f^{(3)}(0) = -1.$

Hence

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Similarly

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

3.  $f(x) = (1 + x)^{-1}$

$$f'(x) = -(1 + x)^{-2}, f^{(2)}(x) = 2(1 + x)^{-3}, f^{(3)}(x) = -2 \cdot 3(1 + x)^{-4}, \dots$$

Generally,  $f^{(n)}(x) = (-1)^n n!(1 + x)^{-n-1}.$

Hence  $f^{(n)}(0) = (-1)^n n!$  and

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \dots, \tag{1.2}$$

which is just the binomial expansion of  $(1 + x)^{-1}.$

### Notes

- The expansion (1.2) is only valid for  $-1 < x < 1.$
- Sometimes the Maclaurin expansion will only be valid for a certain range of  $x$  (in extreme cases, only for  $x = 0$ ).
- Not all functions have Maclaurin expansions (e.g.  $\sqrt{x}$  and  $\ln x.$ )
- Clearly  $f$  must have derivatives of all orders existing at  $x = 0$  in order to have a Maclaurin series (i.e. it must be infinitely differentiable at  $x = 0$ ).
- The first few terms of the Maclaurin expansion of a function can give a good approximation to the function for small values of  $x.$  This can be of great use in finding approximate solutions to equations.

## 1.2 Taylor Series

The Maclaurin expansion is a special case of **Taylor's theorem**, which states that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \quad (1.3)$$

for any value of the constant  $a$ . Choosing  $a = 0$  results in the Maclaurin expansion.

The series (1.3) is referred to as the **Taylor expansion (or series) of  $f$  about  $x = a$** .

An equivalent form of the Taylor expansion of  $f(x)$  about  $x = a$  is

$$f(a+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)x^n}{n!}.$$

Taylor's theorem can be verified in the same way as the Maclaurin series, by differentiating a power series expansion of  $f(x)$  (in powers of  $(x-a)$ ) and setting  $x = a$ .

### Notes

- The functions  $\sqrt{x}$  does not have a Maclaurin series but does have a Taylor series:

$$\sqrt{x} = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16} + \dots$$

- While the function  $\ln x$  does not have a Maclaurin series,  $\ln(1+x)$  does have such an expansion. This is equivalent to the Taylor series of  $\ln x$  about  $x = 1$ :

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

This is valid for  $-1 < x \leq 1$ .

- If  $f$  has a Maclaurin or Taylor series, then that series can always be differentiated (or integrated) any number of times. If the original series is valid for  $|x-a| < k$  then all of the series obtained will also be valid for  $|x-a| < k$ .

### 1.3 L'Hôpital's Rule

When evaluating the derivative of a function from first principles, we have to evaluate the limit of a ratio of functions that both tend to zero. More generally, we often need to evaluate limits of the form

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

when  $f(a) = 0 = g(a)$ . If both  $f(x)$  and  $g(x)$  are differentiable, and  $f(a) = 0 = g(a)$ , l'Hôpital's rule states that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

Note that one differentiates top and bottom **separately**. In particular, if  $g'(a)$  is not zero, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

The rule may be proved by using Taylor's theorem. If  $f(x)$  and  $g(x)$  have a Taylor expansion about  $x = a$ , then since  $f(a) = 0 = g(a)$  we have

$$f(x) = f'(a)(x - a) + \frac{f^{(2)}(a)(x - a)^2}{2!} + \dots$$

$$g(x) = g'(a)(x - a) + \frac{g^{(2)}(a)(x - a)^2}{2!} + \dots$$

and so

$$\frac{f(x)}{g(x)} = \frac{f'(a) + \frac{f^{(2)}(a)(x - a)}{2!} + \dots}{g'(a) + \frac{g^{(2)}(a)(x - a)}{2!} + \dots}.$$

If  $g'(a)$  is not zero, then the result follows.

If  $g'(a) = 0$  then, for the limit to exist,  $f'(a)$  must also be zero and so

$$\frac{f(x)}{g(x)} = \frac{\frac{f^{(2)}(a)}{2!} + \frac{f^{(3)}(a)(x - a)}{3!} + \dots}{\frac{g^{(2)}(a)}{2!} + \frac{g^{(3)}(a)(x - a)}{3!} + \dots}.$$

The validity of the rule can now be seen.

- In practice, one continues to differentiate the numerator and denominator until the expression is no longer of the form  $0/0$ .
- Do **not** differentiate the expression as a quotient. Differentiate the numerator and denominator **separately**.

- The rule also applies when both  $f$  and  $g$  approach  $\pm\infty$  as  $x$  approaches  $a$ .
- The rule also applies when  $x \rightarrow \pm\infty$ , i.e. to limits of the form

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$$

when  $f$  and  $g$  either both approach zero as  $x \rightarrow \infty$  or both approach infinity as  $x \rightarrow \infty$ .

- If the given problem involves the indeterminate form  $0 \times \infty$ , then this can usually be solved by using some preliminary algebra to get the expression into the required form.
- The rule can be applied to the indeterminate forms  $0^0$ ,  $\infty^0$  and  $1^\infty$  by first taking logarithms.

The form  $\infty - \infty$  will certainly require some preliminary algebra to cast it into a form suitable for L'Hôpital's Rule.

### Examples

$$1. \lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$$

$$2. \lim_{x \rightarrow 0} \frac{e^{2x} - 1}{x} = \lim_{x \rightarrow 0} \frac{2e^{2x}}{1} = 2$$

$$3. \lim_{x \rightarrow 1} \frac{2x^3 - 3x^2 + 1}{1 + \cos(\pi x)} = \lim_{x \rightarrow 1} \frac{6x^2 - 6x}{-\pi \sin(\pi x)} = \lim_{x \rightarrow 1} \frac{12x - 6}{-\pi^2 \cos(\pi x)} = \frac{6}{\pi^2}$$

$$4. \lim_{x \rightarrow -\infty} e^x x^2 = \lim_{x \rightarrow -\infty} \frac{x^2}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{2x}{-e^{-x}} = \lim_{x \rightarrow -\infty} \frac{2}{e^{-x}} = 0$$

$$5. \lim_{x \rightarrow 0} \left( \frac{1}{x^2} - \frac{1}{x \sin x} \right) = \lim_{x \rightarrow 0} \frac{\sin x - x}{x^2 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{\cos x - 1}{x^2 \cos x + 2x \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{-\sin x}{-x^2 \sin x + 4x \cos x + 2 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{-\cos x}{-x^2 \cos x - 6x \sin x + 6 \cos x} = -\frac{1}{6}$$

$$6. \lim_{x \rightarrow 0^+} x \ln(\sin x) = \lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{x^{-1}}$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{\sin x} \cdot \cos x}{-x^{-2}}$$

$$= \lim_{x \rightarrow 0^+} \left( -x \cdot \frac{x}{\sin x} \cdot \cos x \right) = -0.1.1 = 0$$