University of Sheffield

School of Mathematics and Statistics

MAS140: Mathematics (Chemical)
MAS152: Civil Engineering Mathematics
MAS152: Essential Mathematical Skills & Techniques
MAS156: Mathematics (Electrical and Aerospace)
MAS161: General Engineering Mathematics

Semester 1 2017–18

Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,
`http://engmaths.group.shef.ac.uk/mas140/`
`http://engmaths.group.shef.ac.uk/mas151/`
`http://engmaths.group.shef.ac.uk/mas152/`
`http://engmaths.group.shef.ac.uk/mas156/`
`http://engmaths.group.shef.ac.uk/mas161/`
which can also be accessed through MOLE.

1 Maclaurin and Taylor Series

1.1 Maclaurin Series

We say that a function $f(x)$ possesses a **power series expansion** if there exist numbers a_0, a_1 etc., which are independent of x, such that

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \tag{1.1}$$

Suppose that $f(x)$ does possess a power series expansion. Then setting $x = 0$ gives $a_0 = f(0)$. If we further assume that the power series expansion can be differentiated, we have

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \dots,$$

and setting $x = 0$ in this expression gives $a_1 = f'(0)$.

Repeating the process (of differentiating and then setting $x = 0$), we have

$$f''(x) = 2a_2 + 3.2a_3 x + \dots \qquad \Rightarrow \qquad a_2 = \frac{f''(0)}{2}$$

$$f'''(x) = 3.2a_3 + 4.3.2a_4 x + \dots \qquad \Rightarrow \qquad a_3 = \frac{f'''(0)}{3.2}$$

$$f^{(4)}(x) = 4.3.2a_4 + \dots \qquad \Rightarrow \qquad a_4 = \frac{f^{(4)}(0)}{4.3.2}.$$

The general coefficient in the power series is

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{for } n = 0, 1, 2, 3, \dots$$

(Recall that 0!=1 and $f^{(n)}$ is the nth derivative of f.)

Having obtained an expression for all the a_n, we can substitute these into Eq. (1.1) to obtain

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f^{(3)}(0)\frac{x^3}{3!} + \dots \quad = \quad \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$$

This is the **Maclaurin series** (or expansion) of f.

Examples

1. $f(x) = e^x \quad \Rightarrow \quad f'(x) = e^x \quad \Rightarrow \quad f^{(n)}(x) = e^x$

   Thus $f^{(n)}(0) = 1$ for $n = 0, 1, 2, \dots$, and so

   $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

2. $f(x) = \sin x$

   $f'(x) = \cos x,\ f^{(2)}(x) = -\sin x,\ f^{(3)}(x) = -\cos x,\ f^{(4)}(x) = \sin x, \ldots$

   Therefore $f(0) = 0,\ f^{(1)}(0) = 1,\ f^{(2)}(0) = 0,\ f^{(3)}(0) = -1.$

   Hence

   $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

   Similarly

   $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

3. $f(x) = (1+x)^{-1}$

   $f'(x) = -(1+x)^{-2}, \quad f^{(2)}(x) = 2(1+x)^{-3}, \quad f^{(3)}(x) = -2.3(1+x)^{-4}, \quad \ldots$

   Generally, $f^{(n)}(x) = (-1)^n n!(1+x)^{-n-1}$.

   Hence $f^{(n)}(0) = (-1)^n n!$ and

   $$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \ldots, \tag{1.2}$$

   which is just the binomial expansion of $(1+x)^{-1}$.

Notes

- The expansion (1.2) is only valid for $-1 < x < 1$.
- Sometimes the Maclaurin expansion will only be valid for a certain range of x (in extreme cases, only for $x = 0$).
- Not all functions have Maclaurin expansions (e.g. $\sqrt{x}$ and $\ln x$.)
- Clearly f must have derivatives of all orders existing at $x = 0$ in order to have a Maclaurin series (i.e. it must be infinitely differentiable at $x = 0$).
- The first few terms of the Maclaurin expansion of a function can give a good approximation to the function for small values of x. This can be of great use in finding approximate solutions to equations.

1.2 Taylor Series

The Maclaurin expansion is a special case of **Taylor's theorem**, which states that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)(x-a)^n}{n!} \tag{1.3}$$

for any value of the constant a. Choosing $a = 0$ results in the Maclaurin expansion.
The series (1.3) is referred to as the **Taylor expansion (or series) of f about $x = a$**.

An equivalent form of the Taylor expansion of $f(x)$ about $x = a$ is

$$f(a+x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)x^n}{n!}.$$

Taylor's theorem can be verified in the same way as the Maclaurin series, by differentiating a power series expansion of $f(x)$ (in powers of $(x-a)$) and setting $x = a$.

Notes

- The functions $\sqrt{x}$ does not have a Maclaurin series but does have a Taylor series:

  $$\sqrt{x} = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16} + \ldots$$

- While the function $\ln x$ does not have a Maclaurin series, $\ln(1+x)$ does have such an expansion. This is equivalent to the Taylor series of $\ln x$ about $x = 1$:

  $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

  This is valid for $-1 < x \leq 1$.

- If f has a Maclaurin or Taylor series, then that series can always be differentiated (or integrated) any number of times. If the original series is valid for $|x-a| < k$ then all of the series obtained will also be valid for $|x-a| < k$.

1.3 L'Hôpital's Rule

When evaluating the derivative of a function from first principles, we have to evaluate the limit of a ratio of functions that both tend to zero. More generally, we often need to evaluate limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

when $f(a) = 0 = g(a)$. If both $f(x)$ and $g(x)$ are differentiable, and $f(a) = 0 = g(a)$, l'Hôpital's rule states that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Note that one differentiates top and bottom **separately**. In particular, if $g'(a)$ is not zero, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

The rule may be proved by using Taylor's theorem. If $f(x)$ and $g(x)$ have a Taylor expansion about $x = a$, then since $f(a) = 0 = g(a)$ we have

$$f(x) = f'(a)(x-a) + \frac{f^{(2)}(a)(x-a)^2}{2!} + \ldots$$

$$g(x) = g'(a)(x-a) + \frac{g^{(2)}(a)(x-a)^2}{2!} + \ldots$$

and so

$$\frac{f(x)}{g(x)} = \frac{f'(a) + \dfrac{f^{(2)}(a)(x-a)}{2!} + \ldots}{g'(a) + \dfrac{g^{(2)}(a)(x-a)}{2!} + \ldots}.$$

If $g'(a)$ is not zero, then the result follows.
If $g'(a) = 0$ then, for the limit to exist, $f'(a)$ must also be zero and so

$$\frac{f(x)}{g(x)} = \frac{\dfrac{f^{(2)}(a)}{2!} + \dfrac{f^{(3)}(a)(x-a)}{3!} + \ldots}{\dfrac{g^{(2)}(a)}{2!} + \dfrac{g^{(3)}(a)(x-a)}{3!} + \ldots}.$$

The validity of the rule can now be seen.

- In practice, one continues to differentiate the numerator and denominator until the expression is no longer of the form $0/0$.

- Do **not** differentiate the expression as a quotient. Differentiate the numerator and denominator **separately**.

- The rule also applies when both f and g approach $\pm\infty$ as x approaches a.

- The rule also applies when $x \to \pm\infty$, i.e. to limits of the form

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

  when f and g *either* both approach zero as $x \to \infty$ *or* both approach infinity as $x \to \infty$.

- If the given problem involves the indeterminate form $0 \times \infty$, then this can usually be solved by using some preliminary algebra to get the expression into the required form.

- The rule can be applied to the indeterminate forms 0^0, ∞^0 and 1^∞ by first taking logarithms.

The form $\infty - \infty$ will certainly require some preliminary algebra to cast it into a form suitable for L'Hôpital's Rule.

Examples

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$

2. $\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{x} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2$

3. $\displaystyle\lim_{x\to 1} \frac{2x^3 - 3x^2 + 1}{1 + \cos(\pi x)} = \lim_{x\to 1} \frac{6x^2 - 6x}{-\pi \sin(\pi x)} = \lim_{x\to 1} \frac{12x - 6}{-\pi^2 \cos(\pi x)} = \frac{6}{\pi^2}$

4. $\displaystyle\lim_{x\to -\infty} e^x x^2 = \lim_{x\to -\infty} \frac{x^2}{e^{-x}} = \lim_{x\to -\infty} \frac{2x}{-e^{-x}} = \lim_{x\to -\infty} \frac{2}{e^{-x}} = 0$

5.
$$\begin{aligned}
\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x \sin x}\right) &= \lim_{x\to 0} \frac{\sin x - x}{x^2 \sin x} \\
&= \lim_{x\to 0} \frac{\cos x - 1}{x^2 \cos x + 2x \sin x} \\
&= \lim_{x\to 0} \frac{-\sin x}{-x^2 \sin x + 4x \cos x + 2 \sin x} \\
&= \lim_{x\to 0} \frac{-\cos x}{-x^2 \cos x - 6x \sin x + 6 \cos x} = -\frac{1}{6}
\end{aligned}$$

6. $\displaystyle\lim_{x\to 0+} x \ln(\sin x) = \lim_{x\to 0+} \frac{\ln(\sin x)}{x^{-1}}$

$$= \lim_{x\to 0+} \frac{\dfrac{1}{\sin x}.\cos x}{-x^{-2}}$$

$$= \lim_{x\to 0+} \left(-x.\frac{x}{\sin x}.\cos x\right) = -0.1.1 = 0$$