University of Sheffield

School of Mathematics and Statistics

MAS140: Mathematics (Chemical)
MAS152: Civil Engineering Mathematics
MAS152: Essential Mathematical Skills & Techniques
MAS156: Mathematics (Electrical and Aerospace)
MAS161: General Engineering Mathematics

Semester 1 2017–18

Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,
`http://engmaths.group.shef.ac.uk/mas140/`
`http://engmaths.group.shef.ac.uk/mas151/`
`http://engmaths.group.shef.ac.uk/mas152/`
`http://engmaths.group.shef.ac.uk/mas156/`
`http://engmaths.group.shef.ac.uk/mas161/`
which can also be accessed through MOLE.

1 Differentiation

A function f is said to be **differentiable** at the point x_0 if f is defined at x_0 (so that $f(x_0)$ exists) and the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \qquad (\star)$$

exists (and so, in particular, is independent of the side from which x approaches x_0). The function f is said to be **differentiable in an interval** if it is differentiable at each point in that interval. At a point at which the limit $(\star)$ does not exist, f is said to be **nondifferentiable**. The value of the limit $(\star)$ is called the **derivative** or **differential coefficient** of f at x_0 and is denoted by $f'(x_0)$, $\dfrac{df(x_0)}{dx}$ or $\mathrm{D}f$. Here D is called the **D-operator**.
A slightly different form of the definition $(\star)$ is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}. \qquad (\dagger)$$

Remarks:

1. The derivative of a function of x is also a function of x and so may be differentiated with respect to x. This new function, denoted by $f''(x)$, $\dfrac{d^2 f}{dx^2}$ or $\mathrm{D}^2 f$, is called the **second derivative** of f. Higher order derivatives are defined similarly.

2. On a curve $y = f(x)$, $f'(x_0)$ gives the slope of the tangent to the curve at the point $x = x_0$. Thus for a function to be differentiable at a point, its curve must have a definite tangent at that point.

3. If $x = f(t)$ gives the displacement of a particle at time t, then $f'(t)$ gives the velocity of the particle and $f''(t)$ is the acceleration of the particle (at time t).

Example
Suppose that $f(x) = x^2$. Then $f(x_0) = x_0^2$ and $f(x_0 + h) = (x_0 + h)^2 = x_0^2 + 2x_0 h + h^2$.
Thus $(\dagger)$ becomes

$$\begin{aligned} f'(x_0) &= \lim_{h \to 0} \frac{(x_0^2 + 2x_0 h + h^2) - x_0^2}{h} \\ &= \lim_{h \to 0} (2x_0 + h) \\ &= 2x_0 \end{aligned}$$

Thus $f(x) = x^2 \Rightarrow f'(x) = 2x$.

1.1 Rules for Differentiation

Suppose that $f(x)$ and $g(x)$ are differentiable functions on some interval and that c is a constant. Then $cf(x)$, $f(x) + g(x)$, $f(x) - g(x)$ are also differentiable and (*i*) $\dfrac{d}{dx}[cf(x)] = c\dfrac{df(x)}{dx}$

(*ii*) $\dfrac{d}{dx}[f(x) \pm g(x)] = \dfrac{df(x)}{dx} \pm \dfrac{dg(x)}{dx}$

Product Rule (*iii*) $\dfrac{d}{dx}[f(x)g(x)] = f(x)\dfrac{dg(x)}{dx} + \dfrac{df(x)}{dx}g(x)$

(*iv*) $\dfrac{d}{dx}\left(\dfrac{1}{g(x)}\right) = -\dfrac{1}{[g(x)]^2}\dfrac{dg(x)}{dx}$

Quotient Rule (*v*) $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{1}{[g(x)]^2}\left(g(x)\dfrac{df(x)}{dx} - f(x)\dfrac{dg(x)}{dx}\right)$

Chain Rule (*vi*) Suppose that $g(x)$ is a differentiable function of x and $x = h(t)$ is a differentiable function of t. Then $f(t) = g[h(t)]$ is a differentiable function of t and

$$\frac{df}{dt} = \frac{dg}{dx}\frac{dh}{dt} \quad \text{or} \quad \frac{df}{dt} = \frac{dg}{dx}\frac{dx}{dt}.$$

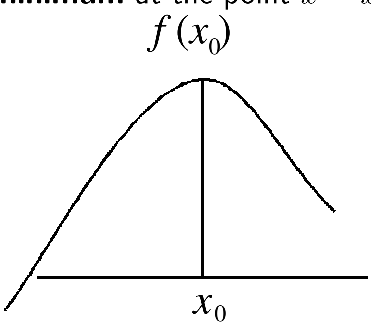
Inverse Function (*vii*) If $y = f(x)$ has an inverse $x = f^{-1}(y)$ then

$$\frac{dy}{dx} = 1 \bigg/ \frac{dx}{dy}$$

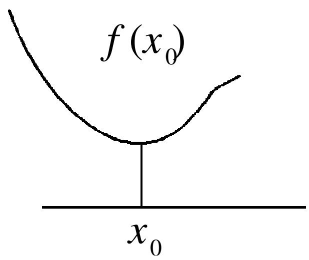
Note that the l.h.s. is a function of x and the r.h.s. is a function of y.
Obviously we require $g(x) \neq 0$ in order for (*iv*) and (*v*) to be valid.

1.2 Maxima and Minima

A function f has a **maximum** at the point $x = x_0$ if $f(x_0) > f(x)$ for all x **near** to x_0.
Likewise it has a **minimum** at the point $x = x_0$ if $f(x_0) < f(x)$ for all x **near** to x_0.



(local) maximum



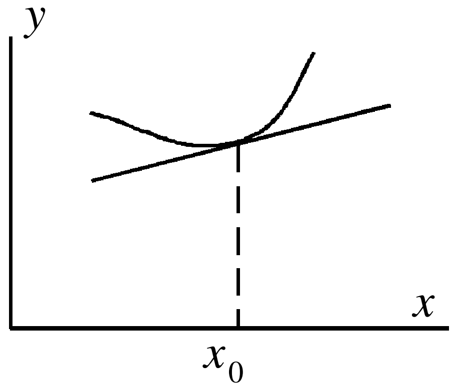
(local) minimum

At a

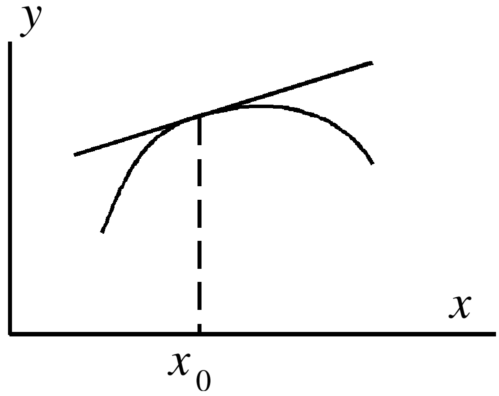
- **maximum** $x < x_0 \Rightarrow f'(x) > 0$ and $x > x_0 \Rightarrow f'(x) < 0$

- **minimum** $x > x_0 \Rightarrow f'(x) < 0$ and $x < x_0 \Rightarrow f'(x) > 0$.

If $f''(x) > 0$ on an interval then $f'(x)$ is increasing and the function f, or the curve $y = f(x)$, is said to **convex** on that interval. Likewise, if $f''(x) < 0$ then the curve is said to be **concave**.



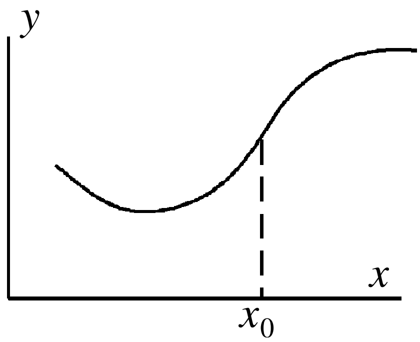
Convex



Concave

When the curve is convex, the curve lies above its tangent and when concave, below its tangent.

A **point of inflexion** is a point at which a curve changes from being convex to concave (or concave to convex) as shown below.



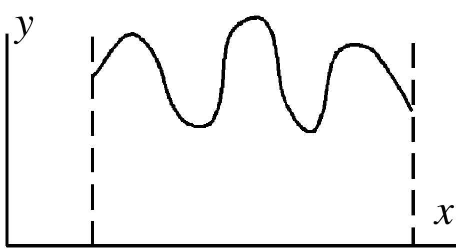
At a point of inflexion there is a change of sign in $f''(x)$.

If x_0 is such that $f'(x_0) = 0$ and $f''(x_0) \neq 0$ then the function $f(x)$ has a

- **maximum** at $x = x_0$ if $f''(x_0) < 0$
- **minimum** at $x = x_0$ if $f''(x_0) > 0$.

There is a point of inflexion at x_0 if $f''(x_0) = 0$ and $f'''(x_0) \neq 0$.

Note: A maximum or minimum refers to **local** behaviour and not to the **greatest or least value** in an interval. A function may have many (local) maxima or minima on an interval but only one greatest value.

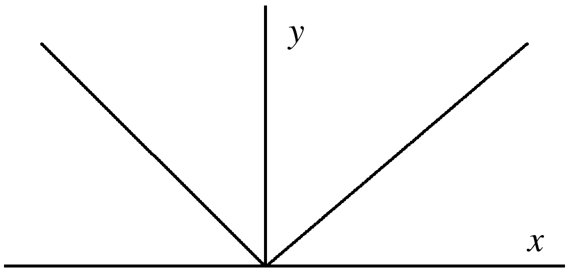


1.3 Continuity and Differentiability

1. A function f which is continuous at $x = x_0$ need not be differentiable at that point. This is shown by the modulus function:

$$f(x) = |x| = \begin{cases} x & (x \geq 0) \\ \\ -x & (x < 0) \end{cases}$$

whose graph is



Clearly, f is continuous everywhere. Also the derivative $f'(x)$ exists everywhere except at $x = 0$. Indeed,

$$\begin{aligned} \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0+} \frac{h}{h} = 1 \\ \lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \to 0-} \frac{-h}{h} = -1 \end{aligned}$$

2. If $f(x)$ is differentiable at $x = x_0$ then f is continuous at x_0. This is because

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad \lim_{x \to x_0} (x - x_0) \quad \text{both exist}$$

$$\Rightarrow \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{(x - x_0)} (x - x_0) \right) \quad \text{exists}$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0).$$

1.4 List of Useful Derivatives

NOTE: It is assumed that x only takes those values for which the functions are defined.

1. $\frac{d}{dx}(constant) = 0$
2. $\frac{d}{dx}x^n = nx^{n-1}$
3. $\frac{d}{dx}\sin x = \cos x$
4. $\frac{d}{dx}\cos x = -\sin x$
5. $\frac{d}{dx}\tan x = \sec^2 x$
6. $\frac{d}{dx}\cot x = -\text{cosec}^2 x$
7. $\frac{d}{dx}\sec x = \sec x \tan x$
8. $\frac{d}{dx}\text{cosec}\, x = -\text{cosec}\, x \cot x$
9. $\frac{d}{dx}e^x = e^x$
10. $\frac{d}{dx}a^x = a^x \ln a$
11. $\frac{d}{dx}\ln x = \frac{1}{x}$
12. $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
13. $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
14. $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
15. $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$
16. $\frac{d}{dx}\sec^{-1} x = \pm\frac{1}{x\sqrt{x^2-1}} \quad \begin{cases} + & \text{if } x > 1 \\ - & \text{if } x < -1 \end{cases}$
17. $\frac{d}{dx}\text{cosec}^{-1} x = \pm\frac{1}{x\sqrt{x^2-1}} \quad \begin{cases} - & \text{if } x > 1 \\ + & \text{if } x < -1 \end{cases}$
18. $\frac{d}{dx}\sinh x = \cosh x$
19. $\frac{d}{dx}\cosh x = \sinh x$
20. $\frac{d}{dx}\tanh x = \text{sech}^2 x$
21. $\frac{d}{dx}\coth x = -\text{cosech}^2 x$
22. $\frac{d}{dx}\text{sech}\, x = -\text{sech}\, x \tanh x$
23. $\frac{d}{dx}\text{cosech}\, x = -\text{cosech}\, x \coth x$
24. $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$
25. $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$
26. $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$
27. $\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2}$
28. $\frac{d}{dx}\text{sech}^{-1} x = \frac{1}{x\sqrt{1-x^2}}$
29. $\frac{d}{dx}\text{cosech}^{-1} x = \frac{-1}{x\sqrt{1+x^2}}$