

MAS140: Mathematics (Chemical)

MAS152: Civil Engineering Mathematics

MAS152: Essential Mathematical Skills & Techniques

MAS156: Mathematics (Electrical and Aerospace)

MAS161: General Engineering Mathematics

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## Semester 1 2016–17

### Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

### Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,

<http://engmaths.group.shef.ac.uk/mas140/>

<http://engmaths.group.shef.ac.uk/mas151/>

<http://engmaths.group.shef.ac.uk/mas152/>

<http://engmaths.group.shef.ac.uk/mas156/>

<http://engmaths.group.shef.ac.uk/mas161/>

which can also be accessed through MOLE.

# 1 Differentiation

A function  $f$  is said to be **differentiable** at the point  $x_0$  if  $f$  is defined at  $x_0$  (so that  $f(x_0)$  exists) and the limit

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad (\star)$$

exists (and so, in particular, is independent of the side from which  $x$  approaches  $x_0$ ). The function  $f$  is said to be **differentiable in an interval** if it is differentiable at each point in that interval. At a point at which the limit  $(\star)$  does not exist,  $f$  is said to be **nondifferentiable**.

The value of the limit  $(\star)$  is called the **derivative** or **differential coefficient** of  $f$  at  $x_0$  and is denoted by  $f'(x_0)$ ,  $\frac{df(x_0)}{dx}$  or  $Df$ . Here  $D$  is called the **D-operator**.

A slightly different form of the definition  $(\star)$  is

$$f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}. \quad (\dagger)$$

Remarks:

1. The derivative of a function of  $x$  is also a function of  $x$  and so may be differentiated with respect to  $x$ . This new function, denoted by  $f''(x)$ ,  $\frac{d^2 f}{dx^2}$  or  $D^2 f$ , is called the **second derivative** of  $f$ . Higher order derivatives are defined similarly.
2. On a curve  $y = f(x)$ ,  $f'(x_0)$  gives the slope of the tangent to the curve at the point  $x = x_0$ . Thus for a function to be differentiable at a point, its curve must have a definite tangent at that point.
3. If  $x = f(t)$  gives the displacement of a particle at time  $t$ , then  $f'(t)$  gives the velocity of the particle and  $f''(t)$  is the acceleration of the particle (at time  $t$ ).

## Example

Suppose that  $f(x) = x^2$ . Then  $f(x_0) = x_0^2$  and  $f(x_0 + h) = (x_0 + h)^2 = x_0^2 + 2x_0h + h^2$ .

Thus  $(\dagger)$  becomes

$$\begin{aligned} f'(x_0) &= \lim_{h \rightarrow 0} \frac{(x_0^2 + 2x_0h + h^2) - x_0^2}{h} \\ &= \lim_{h \rightarrow 0} (2x_0 + h) \\ &= 2x_0 \end{aligned}$$

Thus  $f(x) = x^2 \Rightarrow f'(x) = 2x$ .

## 1.1 Rules for Differentiation

Suppose that  $f(x)$  and  $g(x)$  are differentiable functions on some interval and that  $c$  is a constant. Then  $cf(x)$ ,  $f(x) + g(x)$ ,  $f(x) - g(x)$  are also differentiable and

$$(i) \frac{d}{dx}[cf(x)] = c \frac{df(x)}{dx}$$

$$(ii) \frac{d}{dx}[f(x) \pm g(x)] = \frac{df(x)}{dx} \pm \frac{dg(x)}{dx}$$

**Product Rule**

$$(iii) \frac{d}{dx}[f(x)g(x)] = f(x) \frac{dg(x)}{dx} + \frac{df(x)}{dx} g(x)$$

$$(iv) \frac{d}{dx} \left( \frac{1}{g(x)} \right) = -\frac{1}{[g(x)]^2} \frac{dg(x)}{dx}$$

**Quotient Rule**

$$(v) \frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{1}{[g(x)]^2} \left( g(x) \frac{df(x)}{dx} - f(x) \frac{dg(x)}{dx} \right)$$

**Chain Rule**

(vi) Suppose that  $g(x)$  is a differentiable function of  $x$  and  $x = h(t)$  is a differentiable function of  $t$ . Then  $f(t) = g[h(t)]$  is a differentiable function of  $t$  and

$$\frac{df}{dt} = \frac{dg}{dx} \frac{dh}{dt} \quad \text{or} \quad \frac{df}{dt} = \frac{dg}{dx} \frac{dx}{dt}.$$

**Inverse Function**

(vii) If  $y = f(x)$  has an inverse  $x = f^{-1}(y)$  then

$$\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$$

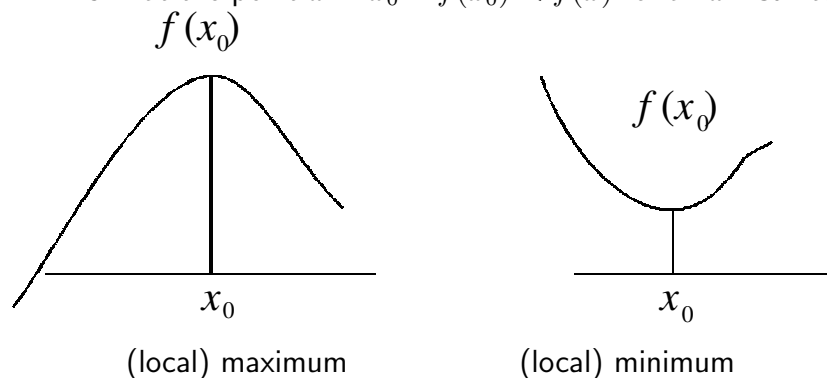
Note that the l.h.s. is a function of  $x$  and the r.h.s. is a function of  $y$ .

Obviously we require  $g(x) \neq 0$  in order for (iv) and (v) to be valid.

## 1.2 Maxima and Minima

A function  $f$  has a **maximum** at the point  $x = x_0$  if  $f(x_0) > f(x)$  for all  $x$  **near** to  $x_0$ .

Likewise it has a **minimum** at the point  $x = x_0$  if  $f(x_0) < f(x)$  for all  $x$  **near** to  $x_0$ .

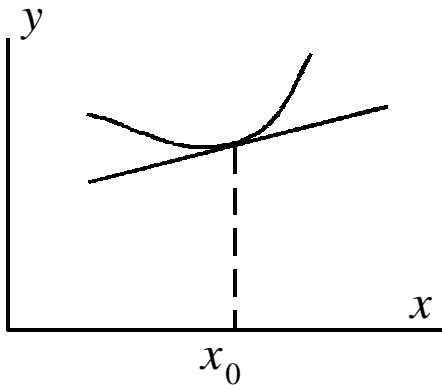


At a

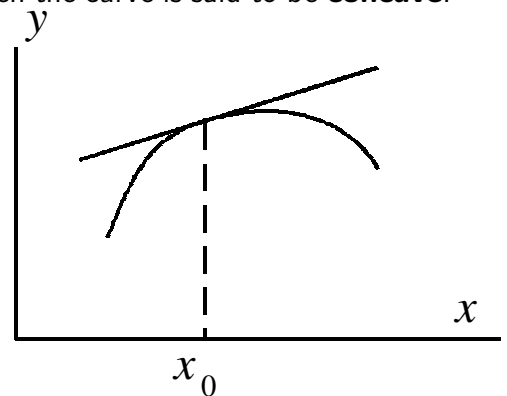
- **maximum**  $x < x_0 \Rightarrow f'(x) > 0$  and  $x > x_0 \Rightarrow f'(x) < 0$

- **minimum**  $x > x_0 \Rightarrow f'(x) < 0$  and  $x < x_0 \Rightarrow f'(x) > 0$ .

If  $f''(x) > 0$  on an interval then  $f'(x)$  is increasing and the function  $f$ , or the curve  $y = f(x)$ , is said to be **convex** on that interval. Likewise, if  $f''(x) < 0$  then the curve is said to be **concave**.



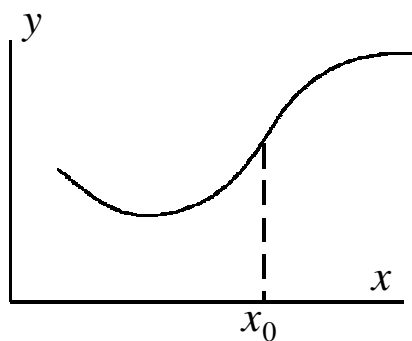
Convex



Concave

When the curve is convex, the curve lies above its tangent and when concave, below its tangent.

A **point of inflexion** is a point at which a curve changes from being convex to concave (or concave to convex) as shown below.



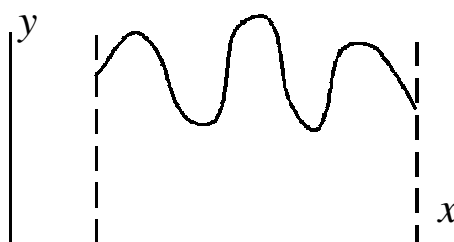
At a point of inflexion there is a change of sign in  $f''(x)$ .

If  $x_0$  is such that  $f'(x_0) = 0$  and  $f''(x_0) \neq 0$  then the function  $f(x)$  has a

- **maximum** at  $x = x_0$  if  $f''(x_0) < 0$
- **minimum** at  $x = x_0$  if  $f''(x_0) > 0$ .

There is a point of inflexion at  $x_0$  if  $f''(x_0) = 0$  and  $f'''(x_0) \neq 0$ .

Note: A maximum or minimum refers to **local** behaviour and not to the **greatest or least value** in an interval. A function may have many (local) maxima or minima on an interval but only one greatest value.

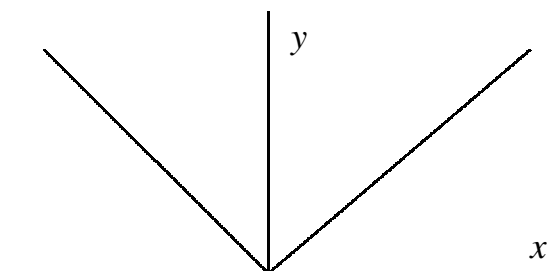


### 1.3 Continuity and Differentiability

1. A function  $f$  which is continuous at  $x = x_0$  need not be differentiable at that point. This is shown by the modulus function:

$$f(x) = |x| = \begin{cases} x & (x \geq 0) \\ -x & (x < 0) \end{cases}$$

whose graph is



Clearly,  $f$  is continuous everywhere. Also the derivative  $f'(x)$  exists everywhere except at  $x = 0$ . Indeed,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^+} \frac{h}{h} = 1 \\ \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0^-} \frac{-h}{h} = -1 \end{aligned}$$

2. If  $f(x)$  is differentiable at  $x = x_0$  then  $f$  is continuous at  $x_0$ . This is because

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \text{and} \quad \lim_{x \rightarrow x_0} (x - x_0) \quad \text{both exist}$$

$$\Rightarrow \lim_{x \rightarrow x_0} \left( \frac{f(x) - f(x_0)}{(x - x_0)} (x - x_0) \right) \quad \text{exists}$$

$$\Rightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0).$$

## 1.4 List of Useful Derivatives

NOTE: It is assumed that  $x$  only takes those values for which the functions are defined.

1.  $\frac{d}{dx}(\text{constant}) = 0$
2.  $\frac{d}{dx}x^n = nx^{n-1}$
3.  $\frac{d}{dx}\sin x = \cos x$
4.  $\frac{d}{dx}\cos x = -\sin x$
5.  $\frac{d}{dx}\tan x = \sec^2 x$
6.  $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$
7.  $\frac{d}{dx}\sec x = \sec x \tan x$
8.  $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$
9.  $\frac{d}{dx}e^x = e^x$
10.  $\frac{d}{dx}a^x = a^x \ln a$
11.  $\frac{d}{dx}\ln x = \frac{1}{x}$
12.  $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$
13.  $\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$
14.  $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
15.  $\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$
16.  $\frac{d}{dx}\sec^{-1} x = \pm \frac{1}{x\sqrt{x^2-1}} \begin{cases} + & \text{if } x > 1 \\ - & \text{if } x < -1 \end{cases}$
17.  $\frac{d}{dx}\operatorname{cosec}^{-1} x = \pm \frac{1}{x\sqrt{x^2-1}} \begin{cases} - & \text{if } x > 1 \\ + & \text{if } x < -1 \end{cases}$
18.  $\frac{d}{dx}\sinh x = \cosh x$
19.  $\frac{d}{dx}\cosh x = \sinh x$
20.  $\frac{d}{dx}\tanh x = \operatorname{sech}^2 x$
21.  $\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$
22.  $\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x$
23.  $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$
24.  $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$
25.  $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$
26.  $\frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}$
27.  $\frac{d}{dx}\coth^{-1} x = \frac{1}{1-x^2}$
28.  $\frac{d}{dx}\operatorname{sech}^{-1} x = \frac{1}{x\sqrt{1-x^2}}$
29.  $\frac{d}{dx}\operatorname{cosech}^{-1} x = \frac{-1}{x\sqrt{1+x^2}}$