University of Sheffield

School of Mathematics and Statistics

MAS140: Mathematics (Chemical)
MAS152: Civil Engineering Mathematics
MAS152: Essential Mathematical Skills & Techniques
MAS156: Mathematics (Electrical and Aerospace)
MAS161: General Engineering Mathematics

Semester 1 2017–18

Outline Syllabus

- **Functions of a real variable.** The concept of a function; odd, even and periodic functions; continuity. Binomial theorem.
- **Elementary functions.** Circular functions and their inverses. Polynomials. Exponential, logarithmic and hyperbolic functions.
- **Differentiation.** Basic rules of differentiation: maxima, minima and curve sketching.
- **Partial differentiation.** First and second derivatives, geometrical interpretation.
- **Series.** Taylor and Maclaurin series, L'Hôpital's rule.
- **Complex numbers.** basic manipulation, Argand diagram, de Moivre's theorem, Euler's relation.
- **Vectors.** Vector algebra, dot and cross products, differentiation.

Module Materials

These notes supplement the video lectures. All course materials, including examples sheets (with worked solutions), are available on the course webpage,
`http://engmaths.group.shef.ac.uk/mas140/`
`http://engmaths.group.shef.ac.uk/mas151/`
`http://engmaths.group.shef.ac.uk/mas152/`
`http://engmaths.group.shef.ac.uk/mas156/`
`http://engmaths.group.shef.ac.uk/mas161/`
which can also be accessed through MOLE.

1 Functions of a real variable

1.1 The concept of a function

Boyle's law for the bulk properties of a gas states that

$$PV = C$$

where P is the pressure, V the volume and C is a constant. Assuming that C is known, if P has a definite value then V is uniquely determined (and likewise P can be found if V is known). We say that V is a function of P (or P is a function of V).

Definitions and Notation

If x and y are two real variables which are related so that y is determined **uniquely** once x is known then we write $y = f(x)$. Any symbol may be used for the function, e.g.

$$y = g(x),\ y = F(x),\ y = y(x).$$

Here, x is called the **independent variable** and y is the **dependent variable**.
The set of all x for which the function f is defined is called the **domain** of f and the set of values y is the **range** (or image set) of f. The value of y when $x = 2$ is denoted by $f(2)$. Strictly one should always distinguish between the name of the function, say f, and the value that it attains at x, which is $f(x)$. In practice, this is very hard to do and we shall often refer to 'the function $f(x)$'.

Examples of functions are

$$x^3 + 5x,\ e^x,\ \ln x,\ \sin x,\ \cos x,\ \sinh x,\ \cosh x.$$

These are examples of **elementary functions**. We shall only be concerned (at this stage) with functions for which the domain and range are (subsets of) real numbers, although the concept of a function can be extended to other types of domain and range.

Graphs of functions

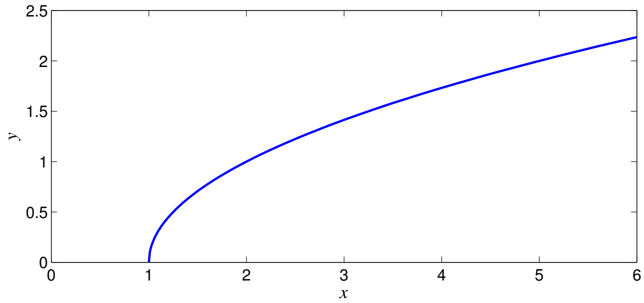
A function f may then be represented graphically by those points with coordinates (x, y) for which $y = f(x)$ and x is in the domain of f. Developing the ability to sketch the principal features of the graph of a function (**curve sketching**) is an important skill that you should practice at every opportunity.
Commonly the domain of a function is an **interval** of the real line. Write (a, b) for the set of numbers x for which $a < x < b$. Note that a and b do not belong to this set. We write $[a, b]$ for the set of numbers x such that $a \leq x \leq b$. Since pressure and density are always positive, the function

$$f(P) = C/P$$

is only defined for $P > 0$ or for the interval $(0, \infty)$. Using this interval notation, the whole real line (i.e. all real numbers) is written as $(-\infty, \infty)$. The restriction of only using real numbers will often imply the domain of a function. For example

$$f(x) = \sqrt{x-1}$$

is only defined for $x \geq 1$ which is the interval $[1, \infty)$. It is to be remembered that $\sqrt{t}$ always means the **positive** square root of t when t is any positive number.



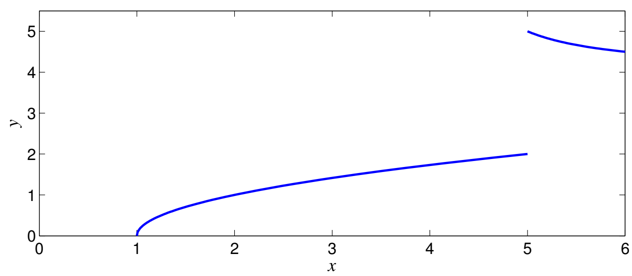
The graph of $y = f(x) = \sqrt{x-1}$

An important property of any function f is that to each value of x in its domain there corresponds precisely one value y in the range. Thus we do not allow $y^2 = x$ to denote a function of x because given any positive value for x there are two possible values of y.

Usually the functions which we shall meet will be defined by a single formula, but this is by no means necessary. For example, consider the function f defined by

$$f(x) = \begin{cases} \sqrt{x-1} & (1 \leq x < 5) \\ 4 + \dfrac{1}{x-4} & (x > 5) \end{cases} \tag{1.1}$$

This function is defined for all $x \geq 1$ except for $x = 5$.



The graph of $y = f(x)$ defined by (1.1)

Note that this is an example of a **discontinuous** function; it has a discontinuity at $x = 5$. We will have more to say about continuity later.

1.2 Even and Odd Functions

A function with the property

$$f(-x) = f(x)$$

is call an **even** function, whereas a function for which

$$f(-x) = -f(x)$$

is called an **odd** function.

Examples

1. $f(x) = x$ is odd since
   $f(-x) = -x = -f(x)$
2. $f(x) = x^2$ is even since
   $f(-x) = (-x)^2 = x^2 = f(x)$
3. $\cos x$ is even and $\sin x$ is odd.

Note that the graph of an even function is symmetric about the y-axis, while the graph of an odd function is rotationally symmetric about the origin.

Most functions are neither odd nor even. For example $f(x) = x + 1$ is neither even nor odd, since $f(-x) = -x + 1$. Note, however, that any function can be written as the sum of an even and an odd function, since the function

$$f_E(x) = \frac{f(x) + f(-x)}{2}$$

is always **even**, while the function

$$f_O(x) = \frac{f(x) - f(-x)}{2}$$

is always **odd**, and

$$f(x) = f_E(x) + f_O(x).$$

1.3 Periodicity

Many phenomena in nature and in engineering are periodic in either time or space, that is, they repeat themselves regularly in time or space. Such phenomena are naturally represented mathematically by **periodic functions**.

A function is said to be **periodic** with period T if

$$f(t) = f(t + nT), \qquad \text{for all integers } n.$$

The **fundamental period** of a periodic function is the smallest positive value of T for which this relationship holds. The most commonly used periodic functions are the trigonometric functions $\cos$, $\sin$ and $\tan$.

1.4 Transformations of functions

Given a function $f(x)$, we can get new functions by transforming the independent variable. For example, we can replace x by $x + a$ for some constant real number a. There are some simple transformations that have simple effects on the graph of the function; these can be useful when sketching the graph of f.

1. *Reversal:* $g(x) = f(-x)$. The graph of $g(x)$ is the reflection in the y-axis of the graph of $f(x)$.
2. *Right shift by* a*:* $g(x) = f(x - a)$. The graph of $g(x)$ is the graph of $f(x)$ translated by amount a along the x-axis.
3. *Left shift by* a*:* $g(x) = f(x + a)$. The graph of $g(x)$ is the graph of $f(x)$ translated by amount $-a$ along the x-axis.
4. *Expansion:* $g(x) = f(ax)$, for $0 < a < 1$. The graph of $g(x)$ is the graph of $f(x)$ expanded (zoomed) by a factor $\dfrac{1}{a}$ along the x-axis.
5. *Contraction:* $g(x) = f(ax)$, for $a > 1$. The graph of $g(x)$ is the graph of $f(x)$ contracted by a factor a along the x-axis.

1.5 Continuity

Suppose a function f is defined on an interval $[a, b]$ which contains c. Then we say that f is **continuous** at c if the graph of f can be drawn on $[a, b]$ without lifting the pen off the paper. While this is not a rigorous mathematical definition of continuity, it captures the essential point about continuity.

The function defined by (1.1) has a discontinuity at $x = 5$, and it can be seen from its graph that it would be necessary to take the pen off the paper at $x = 5$. However, the following function is continuous at every point in its domain:

$$f(x) = \begin{cases} -x & (\infty < x \le 0) \\ x^2 & (0 < x < \infty) \end{cases}$$

1.6 Inverse functions

If $y = f(x)$ is a continuous function of x, then it is sometimes possible to express x as a function of y (if x is uniquely defined whenever y is given). If we can express x as a function of y, i.e. $x = g(y)$, then g is called the **inverse** of f and we write $g = f^{-1}$. Note that this is **not** the same as $\dfrac{1}{f(x)}$. If g is the inverse function of f then f and g satisfy the following relations:

$$f[g(y)] = y \quad \text{and} \quad g[f(x)] = x.$$

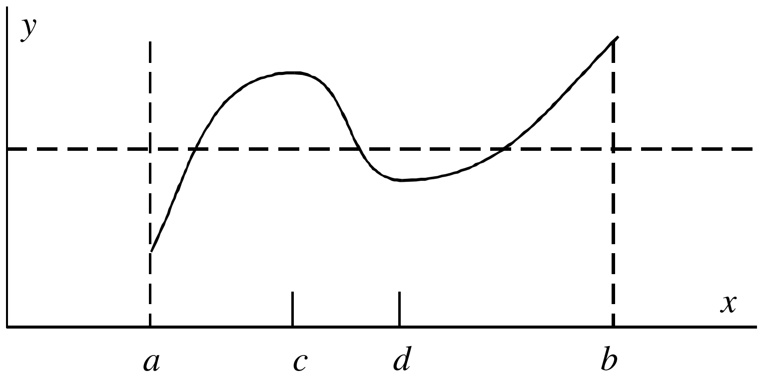
The inverse of $y = f(x)$ will exist if for each value of y there is **just one** value of x. This implies that f is increasing (or decreasing) on its domain. If this is the case, then we can find the inverse of the function f by rearranging the equation $y = f(x)$ to express x in terms of y. Then the resulting expression is $x = f^{-1}(y)$. Note that the domain of f^{-1} is the range of f and its range is the domain of f.

Example: Find the inverse function of $f(x) = 3x - 2$.

Solution: Write $y = f(x) = 3x - 2$. Then

$$y = 3x - 2 \quad \Rightarrow \quad 3x = y + 2 \quad \Rightarrow \quad x = \frac{y+2}{3} = f^{-1}(y)$$

If $f(x)$ is not strictly increasing or decreasing on its domain, then it does not have an inverse on its whole domain. For example, let $y = f(x)$ be defined on $[a, b]$ and have the form shown below.



For some values of y there is more than one corresponding value of x. Thus this function does not have an inverse on the interval $[a, b]$. However, an inverse does exist on each of the intervals $[a, c]$, $[c, d]$ and $[d, b]$, since the function is either strictly increasing or decreasing on these intervals.

Example: The function $f(x) = x^2$ has domain $-\infty < x < \infty$ and range $0 \leq x < \infty$. For any value of $y = f(x)$ in the range of f, there are two corresponding values of x ($\sqrt{x}$ and $-\sqrt{x}$). Therefore, the function $f(x)$ does not have an inverse over its entire domain. However, if we split the domain of $f(x)$, then we can define inverse functions:

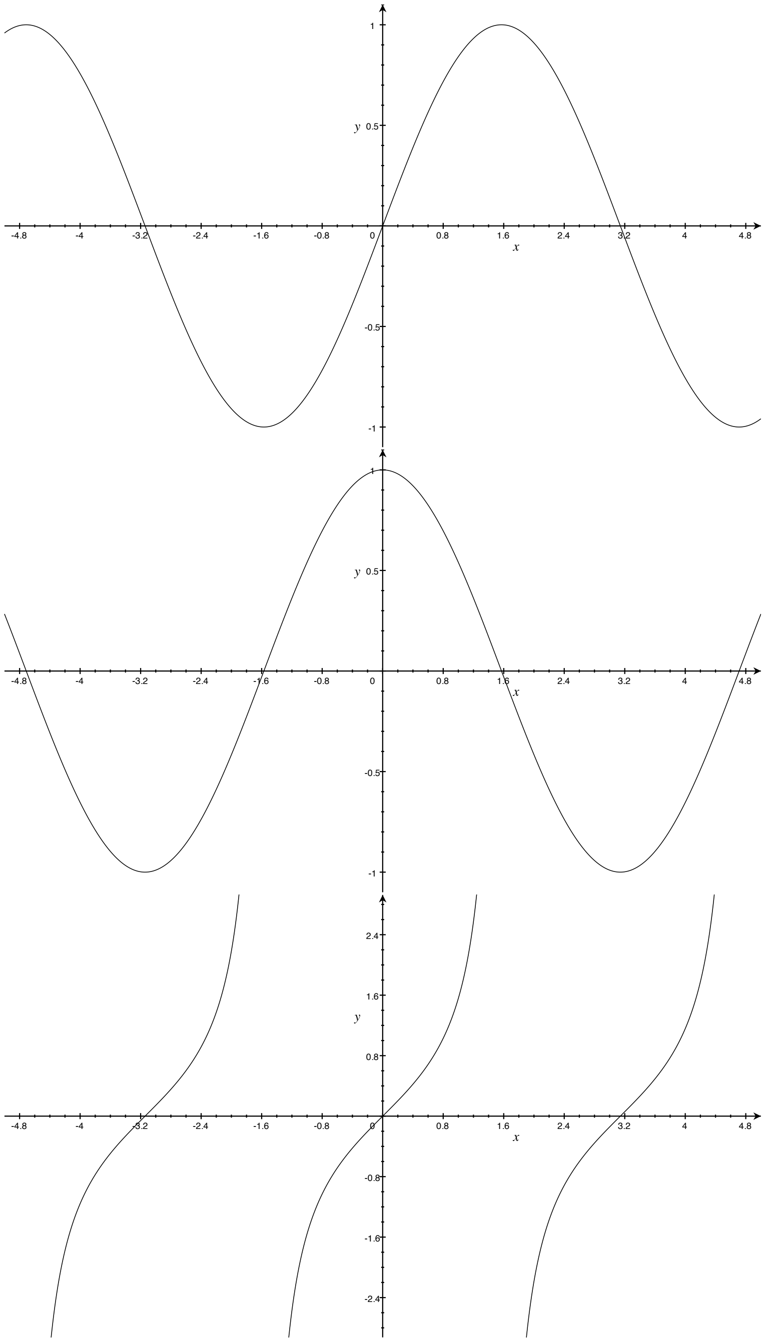
- The function $y = f(x) = x^2$ on domain $0 \leq x < \infty$ has inverse $f^{-1}(y) = \sqrt{y}$.
- The function $y = f(x) = x^2$ on domain $-\infty < x < 0$ has inverse $f^{-1}(y) = -\sqrt{y}$.

Graphs of inverse functions

If a function f has an inverse function f^{-1}, then the graph of f^{-1} is the reflection in the line $y = x$ of the graph of f. Some examples of this can be found in the following section.

1.7 The Circular Functions and their Inverses

The circular functions, or trigonometric functions are a set of elementary functions that arise in trigonometry and the representation of periodic phenomena. The three basic functions are the sine, cosine and tangent functions. For the circular functions, the independent variable is often considered to be an angle, and we express angles in terms of **radians** rather than degrees. An angle of $180°$ corresponds to π radians.
The graphs of the functions $\sin x$, $\cos x$ and $\tan x$ are shown below:



The graphs of $\sin x$ (top), $\cos x$ (middle) and $\tan x$ (bottom).

$\sin x$ and $\cos x$ have domain $-\infty < x < \infty$, while $\tan x$ is not defined whenever x is an odd multiple of $\dfrac{\pi}{2}$; $\sin x$ and $\cos x$ have range $[-1, 1]$, while $\tan x$ has range $(-\infty, \infty)$. All the functions are periodic: $\sin x$ and $\cos x$ have period 2π; $\tan x$ has period π.

The **reciprocal** circular functions (not to be confused with inverse functions) are defined as

$$\sec x = \frac{1}{\cos x}, \quad \mathrm{cosec} x = \frac{1}{\sin x}, \quad \cot x = \frac{1}{\tan x}.$$

Identities

The following identities involving the circular functions prove useful in many applications:

(a) $\cos^2 x + \sin^2 x = 1 \quad \Rightarrow \quad 1 + \tan^2 x = \sec^2 x$

(b) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(c) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

(d) $\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

Inverse circular functions

Because of the periodicity of the circular functions, it is not possible to define inverse functions over the entire domain (this can be seen from the graphs by noting that any line drawn at a constant value of y within the range of the functions crosses the curve of the function more than once). In order to define inverse functions, it is therefore necessary to take a restricted interval of the domain.

The function $y = \sin^{-1} x$ or $y = \arcsin x$ is defined by the value y for which

$$x = \sin y \quad \text{and} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

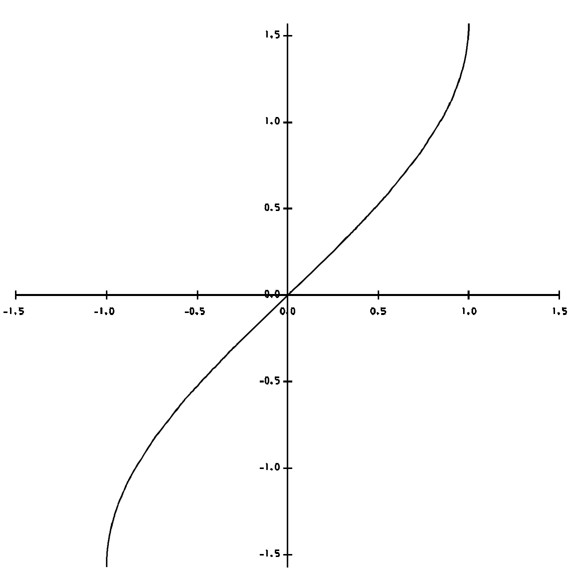
Thus to each value of x in $[-1, 1]$ there corresponds just one value of y in the interval $[-\pi/2, \pi/2]$. Other intervals of y could have been chosen but the one given here is the usual one. Note that there are infinitely many values of y which satisfy $x = \sin y$ for a given value of x. For example, if $x = 1/2$ then $y = n\pi + (-1)^n \pi/6$ for any integer n, but only one of them lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$.

Similarly, $y = \cos^{-1} x$ or $y = \arccos x$ is defined by the value of y for which

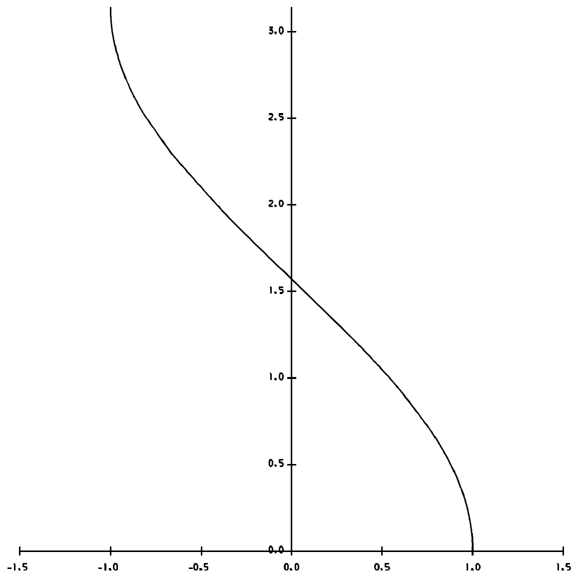
$$x = \cos y \quad \text{and} \quad 0 \le y \le \pi.$$

Note that the interval of y has changed. This is because cos is not monotone (i.e. neither increasing nor decreasing) over the interval $[-\pi/2, \pi/2]$ but it is over the interval $[0, \pi]$. Because we are only concerned with real functions, x is restricted to lie in $[-1, 1]$ for both $\sin^{-1} x$ and $\cos^{-1} x$. The function $y = \tan^{-1} x$ is defined for all (real) values of x, i.e. on $(-\infty, \infty)$ and its range is $(-\pi/2, \pi/2)$.

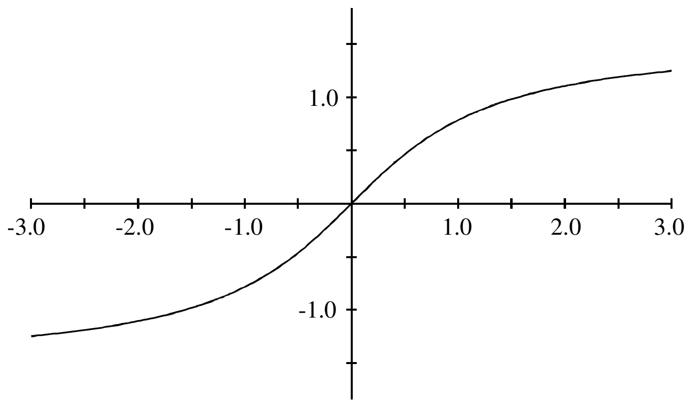
The graphs of $sin^{-1}x$, $\cos^{-1} x$ and $\tan^{-1} x$ are shown below.



$y = \sin^{-1} x$



$y = \cos^{-1} x$



$y = \tan^{-1} x$

Note that the graph of each inverse function is obtained by reflecting the graph of the corresponding function in the line $y = x$.

1.8 Polynomials

Another important class of elementary functions are the **polynomials**. A general polynomial is of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

and has **degree** n (the highest power of x) and $a_0, a_1, \ldots a_n$ are the **coefficients**.

Quadratic functions

A polynomial function of degree 2 is called a **quadratic** function, and has general form

$$Q(x) = a_2 x^2 + a_1 x + a_0.$$

For convenience, we usually write

$$Q(x) = ax^2 + bx + c.$$

By completing the square, it can be shown that $Q(x) = 0$ when

$$x = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac}. \tag{1.2}$$

These values are called the **roots** of $Q(x)$. The roots of $Q(x)$ are both real numbers when $b^2 \geq 4ac$; when this inequality does not hold, the roots are complex numbers (see later).

A quadratic function $Q(x)$ with real roots can be **factorised** into the product of two real linear factors:

$$Q(x) = ax^2 + bx + c = a(x - x_1)(x - x_2),$$

where x_1 and x_2 are the two roots (i.e. the two solutions of (1.2)).

More generally, it can be shown that any polynomial with real coefficients can be factorised as a product of linear and quadratic factors.

1.9 The Binomial Theorem

The **binomial theorem** states that

$$(1 + x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \ldots \tag{1.3}$$

whenever $|x| < 1$, i.e. $-1 < x < 1$ or x lies in the interval $(-1, 1)$, and p is any real number.

- If p is a **positive integer** then the above series is a polynomial of degree p and is valid for all x.
- If p is **not** a positive integer then the above series does not terminate (it is an **infinite series**) and the restriction upon x is necessary to ensure that the series converges.

When p is a positive integer the series may be written as

$$\begin{aligned}(a + x)^n &= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \ldots + \binom{n}{r}a^{n-r}x^r + \ldots + x^n \\ &= \sum_{r=0}^{n}\binom{n}{r}a^{n-r}x^r,\end{aligned}$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = {}^nC_r \tag{1.4}$$

are called the **binomial coefficients**.

$n! = n.(n-1).(n-2)\ldots 3.2.1$, and is referred to as 'n factorial'. This definition only makes sense if n is a positive integer. Note that

$$(n+1)! = (n+1).n!$$

The first few values of $n!$ are $1! = 1,\ 2! = 2,\ 3! = 3.2 = 6,\ 4! = 4.3.2 = 24$. Note that we also define $0! = 1$.

Examples:

1. The binomial expansion of $(1+x)^4$ is

$$\begin{aligned}(1+x)^4 &= 1^4 + \binom{4}{1}1^3.x + \binom{4}{2}1^2.x^2 + \binom{4}{3}1.x^3 + \binom{4}{4}x^4 \\ &= 1 + 4x + 6x^2 + 4x^3 + x^4.\end{aligned}$$

2. The first four terms of the binomial expansion of $(1+x)^{\frac{1}{2}}$, for $-1 < x < 1$, are

$$\begin{aligned}(1+x)^{\frac{1}{2}} &= 1 + \frac{1}{2}x + \frac{1}{2!}.\frac{1}{2}.\left(\frac{1}{2}-1\right)x^2 + \frac{1}{3!}.\frac{1}{2}.\left(\frac{1}{2}-1\right).\left(\frac{1}{2}-2\right)x^3 + \ldots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \ldots.\end{aligned}$$

3. The first three terms of the binomial expansion of $(2-x)^{-1}$ are

$$\begin{aligned}(2-x)^{-1} &= 2^{-1}\left(1-\frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1 + (-1)\left(\frac{-x}{2}\right) + \frac{1}{2!}(-1)(-1-1)\left(\frac{-x}{2}\right)^2 + \ldots\right) \\ &= \frac{1}{2} + \frac{1}{4}x + \frac{1}{8}x^2 + \ldots.\end{aligned}$$

1.10 The exponential and logarithm functions

The **exponential function**, $\exp(x)$ or e^x, is defined for $-\infty < x < \infty$ by the infinite series

$$\exp(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

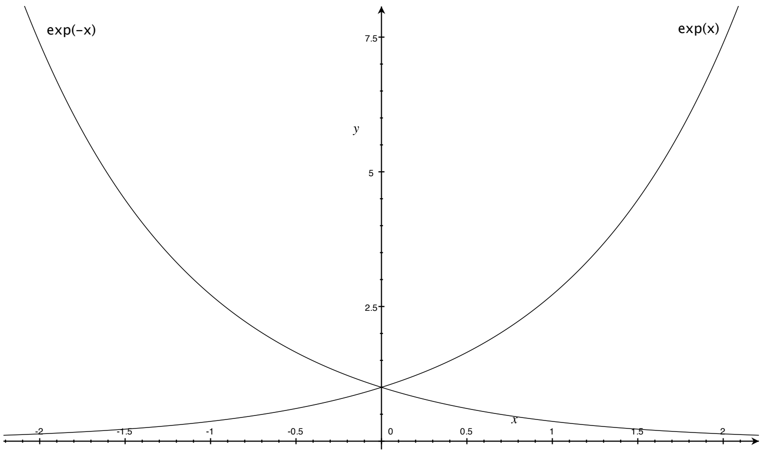
Euler's constant e is defined as $\exp(1)$:

$$e = \exp(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots \simeq 2.71828.$$

Some basic properties of the exponential function:

- $\exp(x+y) = \exp(x).\exp(y)$

- $\exp(-x) = \dfrac{1}{\exp(x)}$
- $\exp(nx) = \exp(x)^n$
- $\exp(0) = 1.$

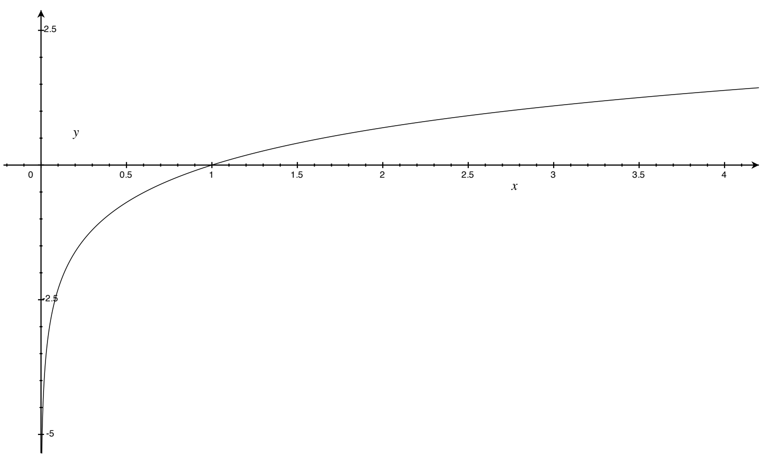


The graphs of $\exp(x)$ and $\exp(-x)$.

The **natural logarithm** function, $\ln(x)$, (also written $\log_e(x)$), is the inverse of the exponential function. Since the range of the exponential function is $(0, \infty)$, $\ln(x)$ is only defined for $x > 0$ (i.e. its domain is $(0, \infty)$). We shall see later that we can use complex numbers to extend the domain of the function.

Some basic properties of the natural logarithm function:

- $\ln(xy) = \ln(x) + \ln(y)$
- $\ln\left(\dfrac{1}{x}\right) = -\ln(x)$
- $\ln(x^n) = n\ln(x)$
- $\ln(1) = 0$
- $\ln(\exp(x)) = x$



The graph of $y = \ln(x)$.

There are also **general logarithm functions**, $\log_a(x)$ (for $a > 0$ and $a \neq 1$). These are defined such that, for fixed a (often referred to as the **base** of the logarithm), the function $\log_a(x)$ is the inverse function of a^x.
In other words, the following equations all mean the same thing:

$$a^x = y, \qquad \log_a(y) = x, \qquad \sqrt[x]{y} = a.$$

This definition justifies the notation $\ln(x) = \log_e(x)$ introduced above.
General logarithms satisfy the following rules:

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a(a^x) = x$ and $a^{\log_a(x)} = x$
- $\log_a(1) = 0$
- $\log_a(1/x) = -\log_a(x)$
- $\log_a(x^r) = r\log_a(x)$
- $\log_a(x) = \ln(x)/\ln(a)$.

The last of these is particularly useful, since it allows us to compute general logarithms using only the natural logarithm.
One commonly seen general logarithm is $\log_{10}$: this measures the number of digits in a number. For example, if $10^5 < x < 10^6$, we have $5 < \log_{10}(x) < 6$, so numbers with logarithms between 5 and 6 are six-digit numbers.

1.11 The hyperbolic functions

The **hyperbolic sine** $(\sinh)$ and the **hyperbolic cosine** $(\cosh)$ are defined by

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \cosh x = \frac{1}{2}(e^x + e^{-x})$$

for $-\infty < x < \infty$.

The hyperbolic tangent $(\tanh)$, cotangent $(\coth)$, cosecant (cosech) and secant (sech) are defined as follows:

$$\tanh x = \frac{\sinh x}{\cosh x}, \quad \coth x = \frac{\cosh x}{\sinh x}$$

$$\text{cosech}x = \frac{1}{\sinh x}, \quad \text{sech}x = \frac{1}{\cosh x}$$

Note that $\sinh x$, $\tanh x$, $\coth x$ and cosech$x$ are all odd functions and that $\cosh x$ and sech$x$ are both even functions.

Identities

The following identities can be proved using the definitions above:

(a) $\cosh x \pm \sinh x = e^{\pm x}$

(b) $\cosh^2 x - \sinh^2 x = 1$

(c) $\text{sech}^2 x = 1 - \tanh^2 x$

(d) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(e) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

These are reminiscent of the identities that hold for the trigonometric functions, and the two sets of functions are indeed related, as we shall see when we study complex numbers.

Series Expansions

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

and so

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \ldots.$$

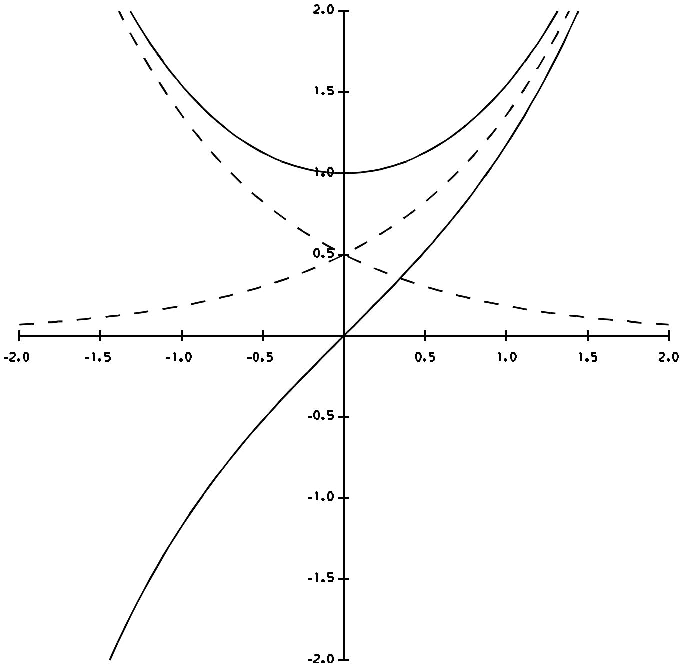
Therefore, we have

$$\begin{aligned}
\sinh x = \tfrac{1}{2}(e^x - e^{-x}) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \\
\cosh x = \tfrac{1}{2}(e^x + e^{-x}) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots
\end{aligned}$$

Note that as we would expect, the series expansion of $\sinh x$ (an odd function) contains only odd powers of x, while that of $\cosh x$ (an even function) contains only even powers of x.

Graphs of Hyperbolic Functions

Since $\cosh x = \frac{1}{2}(e^x + e^{-x})$, when x is large and positive its value is close to $\frac{1}{2}e^x$. Also, when x is large and negative, $\cosh x$ is close to $\frac{1}{2}e^{-x}$.



The graphs of $y = \cosh x$ and $y = \sinh x$, shown as solid lines. The dashed lines show the graphs of e^x and e^{-x}.

The curve $y = \cosh x$ is called the **catenary**. It is the shape of a perfectly flexible chain when hanging freely under gravity.

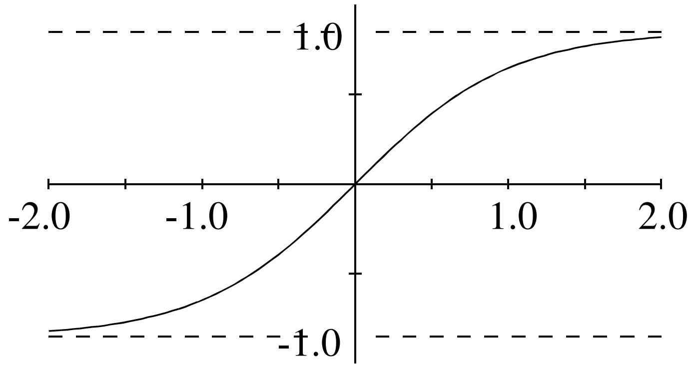
Notes

- $\cosh x \geq 1$ for all $x \in \mathbb{R}$.
- $\cosh x$ decreases when x is negative and increases when x is positive.
- $\sinh x$ takes all real values just once as x increases from $-\infty$ to ∞.
- $\sinh x$ is always increasing.
- When x is large, $\cosh x$ and $\sinh x$ are close to $\frac{1}{2}e^x$

Since $\tanh x = \dfrac{\sinh x}{\cosh x}$, we have

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$$

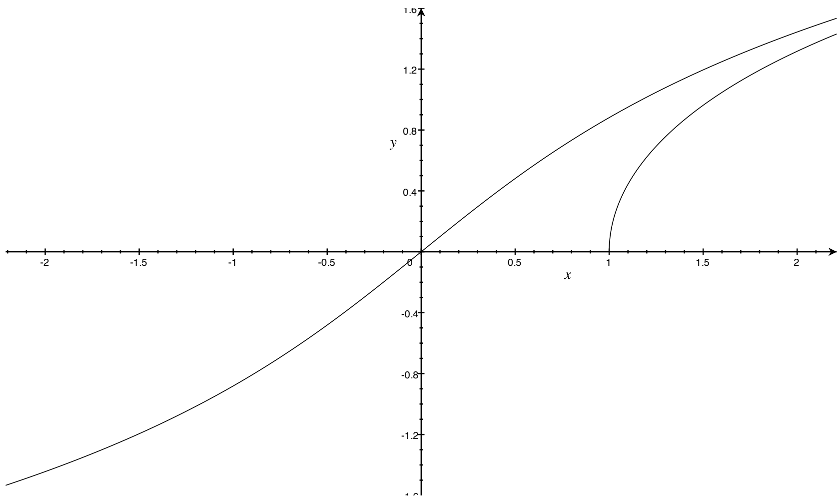
When x is large and positive, $\tanh x$ is close to 1; when x is large and negative, $\tanh x$ is close to -1.



The graph of $y = \tanh x$.

Inverse Hyperbolic Functions

Given any value x there is always a unique number y such that $x = \sinh y$. We call y the inverse hyperbolic sine of x and write $y = \sinh^{-1} x$.



The graphs of $y = \sinh^{-1} x$ (curve passing through the origin) and $y = \cosh^{-1} x$.

In a similar way, we may define inverse functions for all of the hyperbolic functions. Note that sometimes there is a choice to be made for the inverse value (just as we had to for $\sin^{-1}$, etc.) For example, given any $x \geq 1$ we can find a unique y such that $\cosh y = x$ and $y \geq 0$. The graph of the inverse function is always the reflection of the original function in the line $y = x$. However, only part of the graph may be needed.