MAS152 Exercises

Indefinite integration

1. Find the following indefinite integrals:

(a) $\int (1+x)^2 dx$

(b) $\int \frac{1}{x^2} dx$

(c) $\int \frac{1}{(x+1)^2} dx$

(d) $\int \frac{1}{(3x+2)^3} dx$

(e) $\int \frac{1}{x} dx$

(f) $\int \frac{1}{2x+1} dx$

(g) $\int \frac{x}{1+x} dx$

(h) $\int \frac{x}{2+3x} dx$

(i) $\int e^{4x} dx$

(j) $\int \sin 3x \, dx$

(k) $\int \cosh 4x \, dx$

(l) $\int \tan x \, dx$

(m) $\int \sin^2\left(\frac{x}{2}\right) dx$

(n) $\int \cosh^2 3x \, dx$

2. Find the following indefinite integrals, using a suitable substitution if necessary:

(a) $\int t^3 e^{-t^4} dt$

(b) $\int t\cos(t^2-1)\, dt$

(c) $\int x\sqrt{1+x^2}\, dx$

(d) $\int (t+1)\sqrt{t^2+2t}\, dt$

(e) $\int \sin^2 x \cos x \, dx$

(f) $\int \frac{x}{x^2+1} dx$

(g) $\int \frac{3x}{x^2+a^2} dx$

(h) $\int \frac{\cosh u}{1+\sinh^2 u} du$

(i) $\int \frac{\cos t}{\sqrt{1+\sin t}} dt$

(j) $\int \frac{1}{1+9x^2} dx$

(k) $\int \frac{1}{\sqrt{25-16x^2}} dx$

(l) $\int \frac{2x-1}{x^2-x-6} dx$

(m) $\int \sqrt{25-x^2}\, dx$

(n) $\int \frac{dx}{x^2+7x+6}$

(o) $\int \frac{dx}{(x-1)^2(x+1)}$

(p) $\int \frac{x}{(1-x)^2} dx$

(q) $\int \frac{2}{(1-x)(1+x^2)} dx$

(r) $\int \frac{dx}{x^2+10x+30}$

(s) $\int \frac{dx}{\sqrt{x^2-4x+13}}$

(t) $\int \sqrt{x^2+2x-3}\, dx$

(u) $\int \frac{x}{x^2-9} dx$

(v) $\int x^2\sqrt{1-x^2}\, dx$ (Use $x = \sin\theta$)

(w) $\int \frac{4x+6}{x^2+3x+4} dx$

(x) $\int \frac{3x+3}{(x-1)^3(2x+1)} dx$

(y) $\int \frac{\sin x}{\sin x + \cos x} dx$

Integration by parts and definite integration

1. Evaluate the following integrals by parts:

   (a) $\int te^t\,dt$

   (b) $\int x^2 \cosh x dx$

   (c) $\int x^2 \ln x dx$

   (d) $\int \ln x\,dx$ (Hint: let $u = \ln x$, $\frac{dv}{dx} = 1$.)

   (e) $\int \ln(t^2 + a^2)\,dt$

   (f) $\int \cosh^{-1} u\,du$

   (g) $\int \tan^{-1} u\,du$

2. Evaluate the following definite integrals:

   (a) $\int_1^2 \ln x\,dx$

   (b) $\int_{1/2}^{3/4} \frac{dt}{(1-t)(1+2t)}$

   (c) $\int_1^4 \frac{dt}{(1+t)^2}$

   (d) $\int_0^\infty x^2 e^{-ax} dx \quad (a > 0)$

   (e) $\int_0^\infty e^{-2x} \cos 3x\,dx$

   (f) $\int_0^{\ln 2} \sqrt{e^x - 1}\,dx$
   (Hint: let $u = \sqrt{e^x - 1}$.)

3. Which of these integrals are valid? That is, which represent finite, well-defined quantities? Evaluate those that are finite/well-defined, and describe why the others are not.

   (a) $\int_1^\infty \frac{1}{x} dx$

   (b) $\int_0^1 \frac{1}{x} dx$

   (c) $\int_1^\infty x^{-1.1} dx$

   (d) $\int_0^1 x^{-0.9} dx$

   (e) $\int_0^1 x \ln(x) dx$

   (f) $\int_0^\infty \sin(x) e^{-x} dx$

   (g) $\int_{-\infty}^0 e^x dx$

   (h) $\int_0^{\pi/2} \tan x dx$

   (i) $\int_{-1}^1 |x| dx$

   (j) $\int_{-1}^2 \frac{1}{x^3} dx$

Matrices

0. Multiply the matrices together to show that

$$\begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 1 \\ 11 & -3 \end{bmatrix}$$

1. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Compute

(a) AB (b) $(AB)^T$ (c) A^T (d) B^T (e) $B^T A^T$

Hence show that $(AB)^T = B^T A^T$ holds generally in the 2 x 2 case. Explain why this formula remains true if A and B are both n x n.

2. If

$$A = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 0 & 2 \\ 3 & -5 & 1 \\ 2 & 3 & -6 \end{bmatrix}$$

find

(a) $A + 2B$, (b) $2A - B$, (c) $A^T + B^T$, (d) AB, and (e) BA

3. Given that

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(a) Determine which, if any, of X, Y and Z are:
(i) diagonal, (ii) unit matrices, (iii) symmetric, (iv) anti-symmetric

(b) Evaluate

$$X^2 \quad , \quad Y^2 \quad \text{and} \quad Z^2$$

4. Suppose

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$$

Which of the following products are meaningful?

(a) AB, (b) BC, (c) AB^T, (d) AC^T, and (e) BC^T

Evaluate those products that are defined.

Determinants, minors and cofactors

1. Consider the determinant

$$\begin{vmatrix} 3 & -4 & 5 \\ 6 & -5 & -3 \\ -2 & 1 & 2 \end{vmatrix}.$$

   (a) Compute the minors of the elements in the second row.

   (b) Compute the cofactors A_{21}, A_{22}, A_{23} of these elements and hence the value of the determinant.

   (c) Check the value obtained in (ii) by expanding the determinant about its first row.

2. Using the determinant in Question 1, verify the truth of the statement regarding expansion by alien cofactors by evaluating

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23},$$

$$a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23}.$$

3. Show that

$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x + 2a)(x - a)^2.$$

4. Evaluate the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix}.$$

5. Evaluate the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix},$$

   where a, b, c are arbitrary constants. Express your answer as a product of three linear terms.

6. We have seen the theorem: If A, B are both n x n matrices, then $|AB| = |A|\,|B|$. Verify this for the example matrices

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 5 \\ 1 & -4 \end{bmatrix}.$$

Matrix inversion

0. Find the matrix inverse of

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

by applying the 2×2 matrix inverse formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Use the inverse to show that the pair of simultaneous equations $2x+3y+1 = 0 = 5x+7y+3$ has the (unique) solution $x = -2$, $y = 1$.

1. If

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix},$$

find $\mathrm{adj}(A)$, $|A|$, and A^{-1}. Verify that $AA^{-1} = I = A^{-1}A$.

2. If

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},$$

verify that $(AB)^{-1} = B^{-1}A^{-1}$.

3. By matrix methods solve the equations

$$\begin{aligned} 4x - 3y + z &= 11 \\ 2x + y - 4z &= -1 \\ x + 2y - 2z &= 1. \end{aligned}$$

Simultaneous equations

1. Find all values of α such that

$$\begin{aligned} x + 5y + 3z &= 0 \\ 5x + y - \alpha z &= 0 \\ x + 2y + \alpha z &= 0 \end{aligned}$$

has nontrivial solutions. Find these solutions.

2. Show that there is only one value of α for which the equations

$$\begin{aligned} (\alpha + 1)\, x - y + (1 - \alpha)\, z &= 0 \\ 2x + (2 - \alpha)\, y - z &= 0 \\ x + y - z &= 0 \end{aligned}$$

have a non-trivial solution. Find all solutions for this value of α.

3. Show that the three equations

$$\begin{aligned} -2x + y + z &= a \\ x - 2y + z &= b \\ x + y - 2z &= c \end{aligned}$$

have no common solutions unless $a + b + c = 0$, in which case they have infinitely many. Find the solutions when $a = 1, b = 1, c = -2$.

4. Use Gaussian Elimination to

   (a) solve the equations in Question 3 (Week 6),

   (b) invert the matrix in question 1 (Week 6).

5. Find the eigenvalues, along with corresponding eigenvectors, for the following matrices.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$$

Eigenvectors and eigenvalues

The eigenvalues λ of a matrix A are found by solving the **characteristic equation**:

$$|A - \lambda I| = 0,$$

i.e. by finding the roots of the determinant. Here I is the identity matrix. For an $n \times n$ matrix, the characteristic equation has n roots (although they may be complex). The eigenvectors X are found by substituting the eigenvalues into

$$(A - \lambda I)\, X = 0$$

and solving for the components of the column vector X. The normalized eigenvectors $\hat{X}$ are found by dividing by the 'length' of the eigenvector, i.e. $\hat{X} = \frac{1}{|X|}X$, where $|X| = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$.

Questions

1. Let
$$A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$
   (a) Find the eigenvalues of A.
   (b) Find the two normalized eigenvectors, X_1 and X_2.
   (c) Show that the eigenvectors are **orthogonal**: $X_1^T X_2 = 0$.
   (d) Why are the eigenvectors orthogonal?

2. Show that the eigenvalues of the 2×2 antisymmetric matrix
$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$
are *imaginary*, $\lambda = \pm ia$. Show that the same is true of a 3×3 antisymmetric matrix.

3. Find the eigenvalues and corresponding eigenvectors of the following matrices:
   (a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix}$

First-order differential equations

1. Solve the following differential equations:

   (a) $x^3 \dfrac{dy}{dx} = 2x^2 + 3,$

   (b) $x\,(x+1)\,\dfrac{dy}{dx} = y - 1,$

   (c) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2},$

   (d) $(1+x)^2\,\dfrac{dy}{dx} + y^2 = 1,$

   (e) $(y - x)\,\dfrac{dy}{dx} = y$. (Use the substitution $v = y/x$.)

2. Solve the differential equation
$$x^3 \frac{dp}{dx} = a - x.$$
   If $p = 0$ when $x = 2$ and when $x = 4$, find the value of the constant a.

3. Find the solution of the differential equation
$$x^2 \frac{dy}{dx} = y - \frac{dy}{dx}$$
   such that $y = 2$ when $x = \tan(\ln 2)$.

Integrating factors

Solve the following differential equations.

1. $\dfrac{dy}{dx} + 3y = 5\sin 4x$
2. $\dfrac{dy}{dx} + y = e^x,$
3. $\dfrac{dy}{dx} + \dfrac{3y}{x} = x^3,$
4. $\dfrac{dy}{d\theta} + y\cot\theta = \sin\theta,$
5. $\left(1 - x^2\right)\dfrac{dy}{dx} - 1 = xy$
6. $\dfrac{dy}{dx} + \dfrac{4xy}{x^2+1} = \dfrac{1}{\left(x^2+1\right)^2}$
7. $\dfrac{x}{y}\dfrac{dy}{dx} + 1 = xy$ (Use the substitution $v = 1/y$.)
8. $\dfrac{dz}{dx} + 3\dfrac{z}{x} = 2z^2$ (This will require a substitution.)

Second-order differential equations

1. Solve the following differential equations:

   (a) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 6y = 0,$

   (b) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 10y = 0,$

   (c) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0,$

   (d) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$ where $y = 0$ and $\dfrac{dy}{dx} = 2$ when $x = 0$.

2. Find the particular integrals of the following differential equations:

   (a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{5x},$

   (b) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = \sin x,$

   (c) $\dfrac{d^2y}{dx^2} - y = 8xe^{x}.$

3. Solve the following differential equations:

   (a) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{2x},$

   (b) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = xe^{2x},$

   (c) $\dfrac{d^2y}{dx^2} - y = e^{x}\sin x,$

   (d) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = (3x^2 + 2x)e^{x} + 5x^2 - 6x.$

Matrices and differential equations

1. Find the eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}.$$

   Hence construct the solution of the differential equations

$$\frac{dx_1}{dt} = 3x_1 + 4x_2$$

$$\frac{dx_2}{dt} = 4x_1 - 3x_2$$

   such that $x_1 = 1$ and $x_2 = 3$, when $t = 0$.

2. Prove that the characteristic equation associated with the matrix

$$\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

   is given by

$$(1 + \lambda)(1 - \lambda)(2 - \lambda) = 0.$$

   Find the eigenvectors corresponding to the eigenvalues -1, 1 and 2. Hence find the general solution of the differential equations

$$\frac{dx}{dt} = x + y - 2z$$

$$\frac{dy}{dt} = -x + 2y + z$$

$$\frac{dz}{dt} = y - z.$$

Answers

Indefinite integration

1. (a) $\dfrac{(1+x)^3}{3} + c$
   (b) $-\dfrac{1}{x} + c$
   (c) $-\dfrac{1}{x+1} + c$
   (d) $-\dfrac{1}{6(3x+2)^2} + c$
   (e) $\ln|x| + c$
   (f) $\dfrac{1}{2}\ln|2x+1| + c$
   (g) $x - \ln|1+x| + c$
   (h) $\dfrac{x}{3} - \dfrac{2}{9}\ln|3x+2| + c$
   (i) $\dfrac{1}{4}e^{4x} + c$
   (j) $-\dfrac{1}{3}\cos 3x + c$
   (k) $\dfrac{1}{4}\sinh 4x + c$
   (l) $-\ln|\cos x| + c$
   (m) $\dfrac{1}{2}(x - \sin x) + c$
   (n) $\dfrac{1}{12}\sinh 6x + \dfrac{x}{2} + c$

2. (a) $-\dfrac{1}{4}e^{-t^4} + c$
   (b) $\dfrac{1}{2}\sin(t^2-1) + c$
   (c) $\dfrac{1}{3}\left(1+x^2\right)^{3/2} + c$
   (d) $\dfrac{1}{3}\left(t^2+2t\right)^{3/2} + c$
   (e) $\dfrac{1}{3}\sin^3 x + c$
   (f) $\dfrac{1}{2}\ln\left(x^2+1\right) + c$
   (g) $\dfrac{3}{2}\ln\left(x^2+a^2\right) + c$
   (h) $\tan^{-1}(\sinh u) + c$
   (i) $2\sqrt{1+\sin t} + c$
   (j) $\dfrac{1}{3}\tan^{-1}(3x) + c$
   (k) $\dfrac{1}{4}\sin^{-1}\left(\dfrac{4x}{5}\right) + c$
   (l) $\ln|x^2 - x - 6| + c$
   (m) $\dfrac{x}{2}\sqrt{25-x^2} + \dfrac{25}{2}\sin^{-1}\left(\dfrac{x}{5}\right) + c$
   (n) $\dfrac{1}{5}\ln\left(\left|\dfrac{x+1}{x+6}\right|\right) + c$
   (o) $-\dfrac{1}{2(x-1)} + \dfrac{1}{4}\ln\left(\left|\dfrac{x+1}{x-1}\right|\right) + c$
   (p) $\ln|1-x| + \dfrac{1}{1-x} + c$
   (q) $\dfrac{1}{2}\ln\left(1+x^2\right) - \ln|1-x| + \tan^{-1}x + c$
   (r) $\dfrac{\sqrt{5}}{5}\tan^{-1}\left(\dfrac{x+5}{\sqrt{5}}\right) + c$
   (s) $\sinh^{-1}\left(\dfrac{x-2}{3}\right) + c$
   (t) $\dfrac{(x+1)\sqrt{x^2+2x-3}}{2} - 2\cosh^{-1}\left(\dfrac{x+1}{2}\right) + c$
   (u) $\dfrac{1}{2}\ln|x^2-9| + c$
   (v) $\dfrac{1}{8}\left(\sin^{-1}x - x\left(1-2x^2\right)\sqrt{1-x^2}\right) + c$
   (w) $2\ln\left(x^2+3x+4\right) + c$
   (x) $\dfrac{2}{9}\ln\left|\dfrac{x-1}{2x+1}\right| + \dfrac{1}{3(x-1)} - \dfrac{1}{(x-1)^2} + c$
   (y) $-\dfrac{1}{2}\ln|\sin x + \cos x| + \dfrac{x}{2} + c$

Integration by parts and definite integration

1.

(a) $(t-1)e^t + c$

(b) $(x^2+2)\sinh x - 2x\cosh x + c$

(c) $\frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + c$

(d) $x\ln(x) - x + c$

(e) $t\ln(t^2+a^2) - 2t + 2a\tan^{-1}(t/a) + c$

(f) $u\cosh^{-1}(u) - \sqrt{u^2-1} + c$

(g) $u\tan^{-1}(u) - \frac{1}{2}\ln\left(1+u^2\right) + c$

2.

(a) $2\ln 2 - 1$

(b) $\frac{1}{3}\ln(5/2)$

(c) $\frac{3}{10}$

(d) $2/a^3$

(e) $2/13$

(f) $2-\pi/2$.

3.

(a) divergent due to upper limit

(b) divergent due to lower limit

(c) 10

(d) 10

(e) $-1/4$

(f) $1/2$

(g) 1

(h) divergent due to upper limit

(i) 1

(j) ill-defined (can't integrate this across $x=0$), but could argue by symmetry for 3/8.

Matrices

1.

2.

$$A+2B = \begin{bmatrix} 10 & -2 & 7 \\ 5 & -6 & 3 \\ 6 & 9 & -8 \end{bmatrix} , \quad 2A-B = \begin{bmatrix} 0 & -4 & 4 \\ -5 & 13 & 1 \\ 2 & 3 & 14 \end{bmatrix} , \quad A^T+B^T = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 6 \\ 5 & 2 & -2 \end{bmatrix} ,$$

$$AB = \begin{bmatrix} 8 & 19 & -16 \\ 10 & -17 & -4 \\ 25 & -3 & -17 \end{bmatrix} , \quad BA = \begin{bmatrix} 12 & -2 & 20 \\ 13 & -23 & 8 \\ -11 & -10 & -15 \end{bmatrix}$$

3.

	diagonal	unit	symmetric	anti-symmetric
X	no	no	yes	no
Y	no	no	no	yes
Z	yes	no	yes	no

$$X^2 = Y^2 = Z^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. (a) not defined
   (b) $\begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix}$
   (c) $\begin{bmatrix} 9 & 5 \\ 18 & 2 \end{bmatrix}$
   (d) not defined
   (e) not defined

Determinants, minors and cofactors

1. (a) Minors: $\alpha_{21} = -13, \quad \alpha_{22} = 16, \quad \alpha_{23} = -5$
   (b) Cofactors: $A_{ij} = (-1)^{i+j}\alpha_{ij}$ so $A_{21} = 13, \quad A_{22} = 16, \quad A_{23} = 5.$
   (c) Determinant $|A| = -17$.
2. Alien Cofactor Rule: $a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = 0$ if $i \neq j$.
3.
4. abc
5. $(a-b)(b-c)(c-a)$
6. $|A| = 5, \quad |B| = 3, \quad |AB| = 15.$

Matrix inversion

1.

$$\text{adj}A = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}, \quad |A| = 3, \quad A^{-1} = \frac{1}{|A|}\text{adj}A$$

2. $(AB)^{-1} = \frac{1}{3}\begin{bmatrix} 3 & -3 \\ -1 & 2 \end{bmatrix} = B^{-1}A^{-1}$.
3. $x = 3$, $y = 1$, $z = 2$.

Simultaneous equations

1. $\alpha = 1$, $x = \lambda$, $y = -2\lambda$, $z = 3\lambda$.

2. $\alpha = 2$, $x = y = \lambda$, $z = 2\lambda$.

3. $x = y = \lambda$, $z = 1 + \lambda$.

4. (a) $x = 3, \quad y = 1, \quad z = 2.$

   (b) Matrix inverse is

$$\begin{bmatrix} 11/3 & -3 & 1/3 \\ -7/3 & 3 & -2/3 \\ 2/3 & -1 & 1/3 \end{bmatrix}.$$

5. 2x2 case: Eigenvalues $\lambda = 4$ and -1, with corresponding eigenvectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   3x3 case: Eigenvalues $\lambda = -1, -6$ and 5, with corresponding eigenvectors $\begin{bmatrix} 5 \\ 9 \\ -24 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 11 \\ -3 \\ 11 \end{bmatrix}$.

Eigenvectors and eigenvalues

1. (a) $\lambda = -5, 5$

   (b) (i) $\lambda_1 = 5$, $\hat{X}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, (ii) $\lambda_2 = -5$, $\hat{X}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

   (c) $X_1^T X_2 = \frac{1}{5}(2 - 2) = 0$

   (d) The eigenvectors of a real symmetric matrix are orthogonal if the eigenvalues are distinct ($\lambda_1 \neq \lambda_2$).

2. In the 3×3 case, one eigenvalue is zero and the other two are $\lambda = \pm i\sqrt{a^2 + b^2 + c^2}$.

3. (a) $\lambda_1 = 0$, $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda_2 = 2$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or scalar multiples).

   (b) $\lambda_1 = 4$, $X_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\lambda_2 = -1$, $X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

   (c) $\lambda_1 = -3$, $X_1 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, $\lambda_2 = 3$, $X_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$, $\lambda_3 = 9$, $X_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

   (d) $\lambda_1 = 1$, $X_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$, $\lambda_2 = 2$, $X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\lambda_3 = 3$, $X_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

First-order differential equations

1. Here c and $C \geq 0$ are constants.

   (a) $y = 2\ln x - \frac{3}{2}x^{-3} + c$

   (b) $y = \frac{Cx}{x+1} + 1$

   (c) $y = \frac{x+c}{1-cx}$

   (d) $y = \frac{Ce^{-2/(1+x)}-1}{Ce^{-2/(1+x)}+1}$

   (e) $y = x \pm \sqrt{x^2 - C}$

2. $a = 8/3$

3. $y = e^{\tan^{-1} x}$

Integrating factors

1. $y = \frac{1}{5}(3\sin 4x - 4\cos 3x) + ce^{-3x}$

2. $y = \frac{1}{2}e^x + ce^{-x}$

3. $y = \frac{1}{7}x^4 + cx^{-3}$

4. $y = \frac{\theta}{2\sin\theta} - \frac{1}{2}\cos\theta + \frac{c}{\sin\theta}$

5. $y = \frac{\sin^{-1} x + c}{\sqrt{1-x^2}}$

6. $y = \frac{x+c}{(x^2+1)^2}$

7. $y = \frac{1}{x(c - \ln x)}$

8. $z = [x(1 + cx^2)]^{-1}$

Second-order differential equations

1. (a) $y = Ae^{2x} + Be^{3x}$

   (b) $y = e^x (A\cos(3x) + B\sin(3x))$

   (c) $y = (A + Bx)\sin(3x)$

   (d) $y = (A\cos x + B\sin x)e^{-2x}$

2. (a) $y_p = \frac{1}{12}e^{5x}$

   (b) $y_p = \frac{1}{8}(\sin x - \cos x)$

   (c) $y_p = 2x(x-1)e^x$

3. (a) $y = (x + A)e^{2x} + Be^x$

   (b) $y = (x/7 - 5/49)e^{2x} + e^{-x/2}\left(A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

   (c) $y = -\frac{1}{5}(2\cos x + \sin x)e^x + Ae^x + Be^{-x}$

(d) $y = (A + Bx + x^3/3 + x^4/4)\, e^x + 5x^2 + 14x + 18.$

Matrices and differential equations

1. $x_1(t) = 2e^{5t} - e^{-5t}, \qquad x_2(t) = e^{5t} + 2e^{-5t}.$

2. Eigenvectors: $\lambda_1 = 1, X_1 = [3, 2, 1]$, $\lambda_2 = -1, X_2 = [1, 0, 1]$, $\lambda_3 = 2, X_3 = [1, 3, 1]$. General solution:

$$\begin{aligned} x_1(t) &= 3Ae^t + Be^{-t} + Ce^{2t} \\ x_2(t) &= 2Ae^t \qquad\quad + 3Ce^{2t} \\ x_3(t) &= Ae^t + Be^{-t} + Ce^{2t}. \end{aligned}$$