Maths 140-151 Tutorial Problems
Solutions for Weeks 1-2

1(a) $\int(1+x)^2\,dx = \frac{1}{3}(1+x)^3 + c$ ('by inspection')

or, by substitution, let $u = 1+x \Rightarrow du = dx$

$\Rightarrow \int(1+x)^2\,dx = \int u^2\,du = \frac{1}{3}u^3 + c = \frac{1}{3}(1+x)^3 + c$

or, by expansion, $(1+x)^2 = 1+2x+x^2$

$\Rightarrow \int(1+x)^2\,dx = \int(1+2x+x^2)\,dx = x + x^2 + \frac{1}{3}x^3 + c_1 = \frac{1}{3}(x+1)^3 + \underbrace{(c_1 - 1/3)}_{c}$

There is often more than one way to find an integral.
"By inspection" is good once you have some experience, otherwise use a longer method.

(b) $\int \frac{1}{x^2}\,dx = -\frac{1}{x} + c$ (as $\int x^n\,dx = \frac{1}{n+1}x^{n+1}$ for $n \neq -1$)

(c) $\int \frac{1}{(x+1)^2}\,dx = \frac{-1}{x+1} + c$ (use substitution $u = x+1$)

(d) $\int \frac{1}{(3x+2)^3}\,dx = \int \frac{du}{3u^3}$

$u = 3x+2$
$\Rightarrow du = 3\,dx$

$= -\frac{1}{6u^2} + c = \frac{-1}{6(3x+2)^2} + c$

(e) $\int \frac{1}{x}\,dx = \ln|x| + c$ (as $\frac{d}{dx}(\ln x) = 1/x$)

(f) $\int \frac{1}{\underbrace{2x+1}_{u}}\,dx = \int \frac{du}{2u} = \frac{1}{2}\ln|2x+1| + c$

(g) $\int \frac{x}{1+x}\,dx = \int\left(1 - \frac{1}{1+x}\right)dx = x - \ln|x+1| + c$ (use partial fractions) or, make substitution $u = 1+x$.

(h) $\int \frac{x}{2+3x}\,dx = \int\left(\frac{1}{3} - \frac{2}{3(2+3x)}\right)dx = x/3 - \frac{2}{9}\ln|3x+2| + c$ (partial fractions and substitution)

(i) $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$ (use substitution $u = 4x$)

(j) $\int \sin(3x)\,dx = -\frac{1}{3}\cos(3x) + c$ (use substitution $u = 3x$)

(k) $\int \cosh(4x)\,dx = \frac{1}{4}\sinh(4x) + c$ (use substitution $u = 4x$ and recall $\frac{d}{dx}(\sinh x) = \cosh(x)$)

(l) $\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$. Let $u = \cos(x) \Rightarrow du = -\sin(x)\,dx$

$= -\int \frac{du}{u} = -\ln|u| + c = -\ln|\cos(x)| + c$

(m) $\int \sin^2\left(\frac{x}{2}\right)dx = \int \frac{1}{2}(1-\cos x)\,dx$ [use double-angle formula, $\cos(2y) = 1-2\sin^2(y)$ $\Rightarrow \sin^2(y) = \frac{1}{2}(1-\cos 2y)$ with $y = x/2$ here.]

$= x/2 - \frac{1}{2}\sin(x) + c$

(n) $\int \cosh^2(3x)\,dx$. Here we use $\cosh(2y) = \cosh^2(y) + \sinh^2(y) = 2\cosh^2(y) - 1$

$\Rightarrow \cosh^2 y = \frac{1}{2}(1+\cosh(2y))$ [and $y = 3x$ here]

$= \int \frac{1}{2}(1+\cosh(6x))\,dx$

$= \frac{1}{2}\left(x + \frac{1}{6}\sinh(6x)\right) + c = \frac{x}{2} + \frac{1}{12}\sinh(6x) + c$

Maths 140/151 Tutorial Problems

2 (a) $\int t^3 e^{-t^4} dt$. [Let $u = t^4 \Rightarrow du = 4t^3 dt \Rightarrow dt = \frac{1}{4t^3} du$

$= \int \frac{1}{4} e^{-u} du = -\frac{1}{4} e^{-u} + c = -\frac{1}{4} e^{-t^4} + c$

(b) $\int t \cos(t^2 - 1)\, dt$. Let $u = t^2 - 1 \Rightarrow du = 2t\,dt \Rightarrow dt = \frac{1}{2t} du$

$= \int \frac{1}{2} \cos(u)\, du = \frac{1}{2} \sin(u) + c = \frac{1}{2} \sin(t^2 - 1) + c$

(c) $\int x\sqrt{1+x^2}\, dx$. Try $u = 1 + x^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x} du$

$= \int \frac{1}{2} u^{1/2} du = \frac{1}{3} u^{3/2} + c = \frac{1}{3}(1+x^2)^{3/2} + c$

(d) $\int (t+1)\sqrt{t^2+2t}\, dt$. Try $u = t^2 + 2t \Rightarrow du = (2t+2)dt \Rightarrow dt = \frac{1}{2}\frac{du}{(t+1)}$

$= \int \frac{1}{2} u^{1/2} du = \frac{1}{3} u^{3/2} + c = \frac{1}{3}(t^2+2t)^{3/2} + c$

(e) $\int \sin^2 x \cos x\, dx$. Let $u = \sin x \Rightarrow du = \cos(x)\, dx \Rightarrow dx = \frac{du}{\cos(x)}$

$= \int u^2 du = \frac{1}{3} u^3 + c = \frac{1}{3} \sin^3 x + c$

(f) $\int \frac{x}{x^2+1} dx$, let $u = x^2 + 1 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x} du$

$= \int \frac{1}{2u} du = \frac{1}{2} \ln|u| + c = \frac{1}{2} \ln(x^2+1) + c = \ln\sqrt{x^2+1} + c$

(g) $\int \frac{3x}{x^2+a^2} dx$, let $u = x^2 + a^2 \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{1}{2x} du$

$= \int \frac{3}{2u} du = \frac{3}{2} \ln|u| + c = \frac{3}{2} \ln|x^2+a^2| + c = \ln|(x^2+a^2)^{3/2}| + c$

(h) $\int \frac{\cosh(u)\, du}{1+\sinh^2(u)} = \int \frac{du}{\cosh(u)} = \int \frac{2du}{e^u + e^{-u}} = \int \frac{2e^u du}{e^{2u}+1}$. Now make substitution $x = e^u \Rightarrow dx = e^u du$
$\Rightarrow du = e^{-u} dx$

$= \int \frac{2dx}{x^2+1}$. Let $x = \tan(t)$
$\Rightarrow dx = \sec^2(t)\, dt = [1 + \tan^2(t)]dt = (1+x^2)\, dt$

$= 2\int dt = 2\tan^{-1}(e^u) + c$ ①

An alternative (better) way to do this question is to let $\sinh(u) = v$
$\Rightarrow \cosh(u)\, du = dv$
$\Rightarrow \int \frac{dv}{1+v^2}$ and now use $v = \tan(w) \Rightarrow dv = \sec^2(w)\, dw = (1+\tan^2 w)dw$
$= (1+v^2)dw$

$= \int dw = \tan^{-1}(\sinh(u)) + c$ ②

① is equivalent to ②, up to a (different) constant of integration.

(i) $\int \frac{\cos(t)}{\sqrt{1+\sin(t)}} dt$. Let $u = 1 + \sin(t) \Rightarrow du = \cos t\, dt \Rightarrow dt = du/\cos(t)$

$= \int \frac{du}{u^{1/2}} = 2u^{1/2} + c = 2(1+\sin(t))^{1/2} + c = 2\sqrt{1+\sin(t)} + c$

(j) $\int \frac{1}{1+9x^2} dx$. Let $3x = \tan(u) \Rightarrow 3dx = \sec^2 u\, du = (1+\tan^2 u)\, du$
$= (1+9x^2)\, du \Rightarrow dx = \frac{du}{3}(1+9x^2)$

$\Rightarrow \int \frac{du}{3} = \frac{1}{3} u + c = \frac{1}{3} \tan^{-1}(3x) + c$.

(k) $\int \frac{1}{\sqrt{25-16x^2}} dx = \frac{1}{5}\int \frac{dx}{\sqrt{1-(4x/5)^2}}$. Let $\frac{4x}{5} = \sin(u) \Rightarrow dx = \frac{5}{4}\cos(u)\, du$

$= \frac{1}{4}\int \frac{\cos(u)\, du}{\sqrt{1-\sin^2 u}} = \frac{1}{4}\int du = \frac{u}{4} + c = \frac{1}{4} \sin^{-1}\left(\frac{4x}{5}\right) + c$

Maths 140/151 Tutorial Problems

2. (continued) (l) $\int \frac{2x-1}{x^2-x-6}\,dx = \int \frac{2x-1}{(x-3)(x+2)}\,dx = \int \left(\frac{1}{x-3} + \frac{1}{x+2}\right) dx$

$= \ln|x-3| + \ln|x+2| + c = \ln|(x-3)(x+2)| + c = \ln|x^2-x-6| + c$

Alternatively, use the substitution $u = x^2 - x - 6$.

(m) $\int \sqrt{25-x^2}\,dx$. Let $x = 5\sin(t) \Rightarrow dx = 5\cos(t)\,dt$

$= \int 25\cos(t)\sqrt{1-\sin^2(t)}\,dt = 25\int \cos^2(t)\,dt = \frac{25}{2}\int [1+\cos(2t)]\,dt$

$= \frac{25}{2}\left(t + \frac{1}{2}\sin(2t)\right) = \frac{25}{2}\left(\sin^{-1}(x/5) + \frac{x}{5}\sqrt{1-x^2/25}\right) + c$

$= \frac{25}{2}\sin^{-1}\left(\frac{x}{5}\right) + \frac{x}{2}\sqrt{25-x^2} + c$

(n) $\int \frac{dx}{x^2+7x+6} = \int \frac{dx}{(x+6)(x+1)} = \frac{1}{5}\int dx \left(\frac{1}{x+1} - \frac{1}{x+6}\right) = \frac{1}{5}(\ln|x+1| - \ln|x+6|) + c$

$= \frac{1}{5}\ln\left|\frac{x+1}{x+6}\right| + c$

(o) $\int \frac{dx}{(x-1)^2(x+1)} = \int \left(\frac{Ax+B}{(x-1)^2} + \frac{C}{x+1}\right) dx = \int \frac{(A+C)x^2 + (A+B-2C)x + (B+C)}{(x-1)^2(x+1)}\,dx$

$\Rightarrow A+C=0,\quad A+B-2C=0,\quad B+C=1 \quad \Rightarrow B = 3C \Rightarrow C = 1/4,\ A = -1/4,\ B = 3/4$

$\Rightarrow \int \left(\frac{-x+3}{4(x-1)^2} + \frac{1}{4(x+1)}\right) dx = \frac{1}{4}\int \left(-\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{x+1}\right) dx = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{2(x-1)} + c$

It's much easier to do this question with a substitution, $u = x-1$

$\int \frac{du}{u^2(u+2)} = \int \left(\frac{\alpha}{u^2} + \frac{\beta}{u} + \frac{\gamma}{u+2}\right) du \Rightarrow \int \frac{\alpha(u+2) + \beta u(u+2) + \gamma u^2}{u^2(u+2)}\,du$

so that $\beta + \gamma = 0,\quad \alpha + 2\beta = 0,\quad 2\alpha = 1 \Rightarrow \alpha = 1/2,\ \beta = -1/4,\ \gamma = 1/4$

$\Rightarrow -\frac{1}{2u} - \frac{1}{4}\ln|u| + \frac{1}{4}\ln|u+2| = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{1}{2(x-1)} + c$

(p) $\int \frac{x}{(1-x)^2}\,dx = \int \frac{-(1-x)+1}{(x-1)^2}\,dx = \ln|x-1| - \frac{1}{x-1} + c = \ln|1-x| + \frac{1}{1-x} + c$

(q) $\int \frac{2}{(1-x)(1+x^2)}\,dx = \int \left(\frac{A}{1-x} + \frac{Bx+C}{1+x^2}\right) dx = \int \frac{(A-B)x^2 + (B-C)x + A+C}{(1-x)(1+x^2)}\,dx$

$\Rightarrow A-B=0,\quad B-C=0,\quad A+C=2$

$\Rightarrow A = B = C = 1$

$= \int \left(-\frac{1}{x-1} + \frac{x}{1+x^2} + \frac{1}{1+x^2}\right) dx$

$= -\ln|x-1| + \frac{1}{2}\ln|1+x^2| + \tan^{-1}[x] + c$

$= -\ln|1-x| + \frac{1}{2}\ln|1+x^2| + \tan^{-1}(x) + c$

(r) $\int \frac{dx}{x^2+10x+30} = \int \frac{dx}{(x+5)^2+5}$

Let $\sqrt{5}\tan(u) = x+5 \quad \Rightarrow dx = \sqrt{5}\,du\,(1+\tan^2 u)$

$= \int \frac{1}{\sqrt{5}}\,du = \frac{1}{\sqrt{5}}u + c = \frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{x+5}{\sqrt{5}}\right) + c$

(s) $\int \frac{dx}{\sqrt{x^2-4x+13}} = \int \frac{dx}{\sqrt{(x-2)^2+9}}$. Let $x-2 = 3\sinh(u)$

$\Rightarrow dx = 3\cosh(u)\,du$

$= \int \frac{\cosh(u)\,du}{\sqrt{\sinh^2 u + 1}} = \int du = u + c = \sinh^{-1}\left(\frac{x-2}{3}\right) + c$

Maths 140/151 Tutorial Problems

2. (continued)

(t) $\int \sqrt{x^2+2x-3}\,dx = \int \sqrt{(x+1)^2-4}\,dx$. Let $x+1 = 2\cosh(u)$
$\Rightarrow dx = 2\sinh(u)\,du$

$= \int 4\sinh(u)\sqrt{\cosh^2 u - 1}\,du = 4\int \sinh^2(u)\,du$

Now use $\cosh(2u) = 1 + 2\sinh^2(u) \Rightarrow 2\sinh^2(u) = \cosh(2u) - 1$

$= 2\int (\cosh 2u - 1)\,du = \sinh(2u) - 2u + c = 2\sinh(u)\cosh(u) - 2u + c$

$= (x+1)\sqrt{\frac{(x+1)^2}{4} - 1} - 2\cosh^{-1}\left(\frac{x+1}{2}\right) + c$

$= \frac{1}{2}(x+1)\sqrt{x^2+2x-3} - 2\cosh^{-1}\left(\frac{x+1}{2}\right) + C$.

(u) $\int \frac{x}{x^2-9}\,dx$. Let $u = x^2 - 9 \Rightarrow du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x^2-9| + c$.

(v) $\int x^2\sqrt{1-x^2}\,dx$. Let $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$= \int \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int \sin^2(2\theta)\,d\theta = \frac{1}{4}\int \frac{1}{2}(1-\cos(4\theta))\,d\theta$

$= \theta/8 - \frac{1}{32}\sin(4\theta) + c = \frac{1}{8}\sin^{-1}(x) - \frac{1}{16}\sin(2\theta)\cos(2\theta) + c$

$= \frac{1}{8}\sin^{-1}(x) - \frac{1}{8}x\sqrt{1-x^2}(1-2x^2) + c$

(w) $\int \frac{4x+6}{x^2+3x+4}\,dx$. Let $u = x^2+3x+4 \Rightarrow du = (2x+3)\,dx \Rightarrow dx = du/(2x+3)$

$= \int \frac{2}{u}\,du = 2\ln|x^2+3x+4| + c$.

(x) $\int \frac{3x+3}{(x-1)^3(2x+1)}\,dx = \int \frac{3u+6}{u^3(2u+3)}\,du$ where $u = x-1$

$= \int \left(\frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{2u+3}\right)du$

$= \int \frac{(2u^3+3u^2)A + (2u^2+3u)B + (2u+3)C + u^3 D}{u^3(2u+3)}\,du$

$\Rightarrow$ $2A + D = 0$, $3A + 2B = 0$, $3B + 2C = 3$, $3C = 6$
$D = -4/9$ $\quad\Rightarrow A = 2/9$ $\quad\Rightarrow B = -1/3$ $\quad\Rightarrow C = 2$

$\Rightarrow \frac{2}{9}\ln|x-1| + \frac{1}{3(x-1)} - \frac{1}{(x-1)^2} - \frac{2}{9}\ln|2x+1| + c$

(y) $\int \frac{\sin(x)}{\sin(x)+\cos(x)}\,dx$.

$\sin(x) = \frac{1}{2}(\sin x + \cos x) + \frac{1}{2}(\sin x - \cos x)$

$= \int \left(\frac{1}{2} + \frac{1}{2}\frac{\sin(x)-\cos(x)}{\sin(x)+\cos(x)}\right)dx = \frac{x}{2} - \frac{1}{2}\int \frac{du}{u}$ where $u = \sin x + \cos x$

$= \frac{x}{2} - \frac{1}{2}\ln|\sin(x)+\cos(x)| + c$

Maths 140/151 Tutorial Problems

Ex. Sheet 2: Integration / Week 3

1. Integration by parts: $\int uv'\,dx = [uv] - \int \frac{du}{dx} v\,dx$

(a) $\int te^t\,dt$. Let $u = t$, $v' = e^t \Rightarrow v = e^t$, $u' = 1$

$= te^t - \int e^t\,dt = (t-1)e^t + c$

(b) $\int x^2\cosh(x)\,dx = x^2\sinh(x) - \int 2x\sinh(x)\,dx$

$= x^2\sinh(x) - 2x\cosh(x) + 2\int\cosh(x)\,dx$

$= (x^2+2)\sinh(x) - 2x\cosh(x) + c$

(c) $\int \underbrace{x^2}_{v'}\underbrace{\ln(x)}_{u}\,dx = \frac{1}{3}x^3\ln(x) - \int \frac{1}{3}x^3 \times \frac{1}{x}\,dx$

$= \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3 + c$

(d) $\int \ln(x)\,dx = x\ln(x) - \int x \times 1/x\,dx = x(\ln(x)-1) + c$

(e) $\int \ln(t^2+a^2)\,dt = t\ln(t^2+a^2) - \int t \times \frac{2t}{t^2+a^2}\,dt$

$= t\ln(t^2+a^2) - 2\int\left(\frac{(t^2+a^2)-a^2}{t^2+a^2}\right)dt$

$= t\ln(t^2+a^2) - 2t + 2a^2 \times \frac{1}{a}\tan^{-1}(t/a) + c$

$= t\ln(t^2+a^2) - 2t + 2a\tan^{-1}(t/a) + c$

(f) $\int \cosh^{-1}(u)\,du$. Let $u = \cosh(x) \Rightarrow du = \sinh(x)\,dx$

$= \int x\sinh(x)\,dx = x\cosh(x) - \int\cosh(x)\,dx = x\cosh(x) - \sinh(x) + c$

$= x\cosh(x) - \sqrt{\cosh^2(x) - 1} + c$

$= u\cosh^{-1}(u) - \sqrt{u^2-1} + c$

(g) $\int \tan^{-1}(u)\,du$. Let $u = \tan(x) \Rightarrow du = \sec^2(x)\,dx$

$= \int x\sec^2(x)\,dx = x\tan(x) - \int \tan(x)\,dx$

$= x\tan(x) + \ln|\cos(x)| + c = x\tan(x) - \frac{1}{2}\ln|\sec^2(x)| + c$

$= x\tan(x) - \frac{1}{2}\ln|1+\tan^2(x)| + c$

$= u\tan^{-1}(u) - \frac{1}{2}\ln|1+u^2| + c$

Ex. Sheet 2 (Integration) / Week 3

2. Definite integration.

(a) $\int_1^2 \ln(x)\,dx = [x\ln x]_1^2 - \int_1^2 x \times 1/x\,dx = [x(\ln(x)-1)]_1^2$

$= 2(\ln|2|-1) - 1(\ln 1 - 1) = 2\ln 2 - 2 - 0 + 1$

$= 2\ln 2 - 1.$

(b) $\int_{1/2}^{3/4} \frac{dt}{(1-t)(1+2t)} = \int_{1/2}^{3/4}\left(\frac{A}{1-t} + \frac{B}{1+2t}\right)dt$ (partial fractions)

$= \int_{1/2}^{3/4} \frac{A(1+2t) + B(1-t)}{(1-t)(1+2t)}\,dt$ $\Rightarrow$ $A + B = 1$ (t^0); $2A - B = 0$ (t^1)

$\Rightarrow B = 2A,\ A = 1/3,\ B = 2/3$

$= \int_{1/2}^{3/4}\left(\frac{1}{3(1-t)} + \frac{2}{3(1+2t)}\right)dt$

$= \left[-\frac{1}{3}\ln|1-t| + \frac{1}{3}\ln|t+1/2|\right]_{1/2}^{3/4} = \frac{1}{3}\left[\ln\left|\frac{t+1/2}{1-t}\right|\right]_{1/2}^{3/4}$

$= \frac{1}{3}\times\left(\ln\left|\frac{5/4}{1/4}\right| - \ln\left|\frac{1}{1/2}\right|\right) = \frac{1}{3}(\ln(5) - \ln(2))$

(c) $\int_1^4 \frac{dt}{(1+t)^2}$. Let $u = 1+t \Rightarrow du = dt$

$= \int_{t=1}^{4} \frac{du}{u^2} = \left[-\frac{1}{u}\right]_{t=1}^{4} = \left[-\frac{1}{1+t}\right]_1^4 = -\frac{1}{5} + \frac{1}{2} = \frac{5-2}{10} = \frac{3}{10}$

(d) $\int_0^\infty x^2 e^{-ax}\,dx$ $(a>0)$

$= \left[-\frac{1}{a}x^2e^{-ax}\right]_0^\infty + \frac{1}{a}\int 2xe^{-ax}\,dx = \left[\left(-\frac{x^2}{a} - \frac{2x}{a^2}\right)e^{-ax}\right]_0^\infty + \frac{2}{a^2}\int_0^\infty e^{-ax}\,dx$

$= \left[-\frac{1}{a}(x^2 + 2x/a + 2/a^2)\,e^{-ax}\right]_0^\infty$

$= 0 + \frac{1}{a}(0 + 0 + 2/a^2)\times 1 = 2/a^3$

(e) $\int_0^\infty e^{-2x}\cos(3x)\,dx = I$. Let $u = \cos(3x)$, $v' = e^{-2x}$

$I = \left[-\frac{1}{2}e^{-2x}\cos(3x)\right]_0^\infty - \frac{3}{2}\int_0^\infty e^{-2x}\sin(3x)\,dx$

$= \left[\left(-\frac{1}{2}\cos(3x) + \frac{3}{4}\sin(3x)\right)e^{-2x}\right]_0^\infty - \frac{9}{4}\int_0^\infty e^{-2x}\cos(3x)\,dx$

$= 0 - \left(-\frac{1}{2} + 0\right) - \frac{9}{4}I = \frac{1}{2} - \frac{9}{4}I$

$\Rightarrow \frac{13}{4}I = \frac{1}{2} \Rightarrow I = 2/13$

(f) $\int_0^{\ln 2}\sqrt{e^x - 1}\,dx$. let $u = \sqrt{e^x-1} \Rightarrow du = \frac{1}{2}e^x/u\,dx \Rightarrow dx = \frac{2u\,du}{e^x}$

$= \int_0^1 2u^2e^{-x}\,du = \int_0^1 \frac{2u^2}{u^2+1}\,du = \int_0^1\left(2 - \frac{2}{u^2+1}\right)du$

$= \left[2u - 2\tan^{-1}u\right]_0^1 = 2 - 2\tan^{-1}(1) = 2 - 2\times\pi/4 = 2 - \pi/2$

3. (a) Divergent due to upper limit.

(b) Divergent due to lower limit.

(c) 10

(d) 10

(e) $-1/4$

(f) $1/2$

(g) 1

(h) Divergent due to upper limit.

(i) 1

(j) Ill-defined (can't integrate this across $x=0$), but could argue by symmetry for $3/8$.

Week 4: Matrices.

1. $\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

(a) $\underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

(b) $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

(c) $\underline{\underline{A}}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

(d) $\underline{\underline{B}}^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}$

(e) $\underline{\underline{B}}^T\underline{\underline{A}}^T = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{21}b_{11} + a_{22}b_{21} \\ a_{11}b_{12} + a_{12}b_{22} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

(b) = (e), hence $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T\underline{\underline{A}}^T$ in 2D case.

In $n \times n$ case, $A_{ij} = (A^T)_{ji}$

and $(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}$ $\Rightarrow ((AB)^T)_{ij} = \sum_k A_{jk}B_{ki} = \sum_k (A^T)_{kj}(B^T)_{ik}$

$= \sum_k (B^T)_{ik}(A^T)_{kj} = (B^TA^T)_{ij}$

Hence $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T\underline{\underline{A}}^T$

for any $n \times n$ matrices $\underline{\underline{A}}, \underline{\underline{B}}$.

2. $\underline{\underline{A}} = \begin{bmatrix} 2 & -2 & 3 \\ -1 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 4 & 0 & 2 \\ 3 & -5 & 1 \\ 2 & 3 & -6 \end{bmatrix}$

(a) $\underline{\underline{A}} + 2\underline{\underline{B}} = \begin{bmatrix} 10 & -2 & 7 \\ 5 & -6 & 3 \\ 6 & 9 & -8 \end{bmatrix}$, (b) $2\underline{\underline{A}} - \underline{\underline{B}} = \begin{bmatrix} 0 & -4 & 4 \\ -5 & 13 & 1 \\ 2 & 3 & 14 \end{bmatrix}$

(c) $\underline{\underline{A}}^T + \underline{\underline{B}}^T = \begin{bmatrix} 2 & -1 & 2 \\ -2 & 4 & 3 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 3 & 2 \\ 0 & -5 & 3 \\ 2 & 1 & -6 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 4 \\ -2 & -1 & 6 \\ 5 & 2 & -2 \end{bmatrix}$

(d) $\underline{\underline{A}}\,\underline{\underline{B}} = \begin{bmatrix} 8-6+6 & 0+10+9 & 4-2-18 \\ -4+12+2 & 0-20+3 & -2+4-6 \\ 8+9+8 & 0-15+12 & 4+3-24 \end{bmatrix} = \begin{bmatrix} 8 & 19 & -16 \\ 10 & -17 & -4 \\ 25 & -3 & -17 \end{bmatrix}$

(e) $\underline{\underline{B}}\,\underline{\underline{A}} = \begin{bmatrix} 4 & 0 & 2 \\ 3 & -5 & 1 \\ 2 & 3 & -6 \end{bmatrix}\begin{bmatrix} 2 & -2 & 3 \\ -1 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 8+0+4 & -8+0+6 & 12+0+8 \\ 6+5+2 & -6-20+3 & 9-5+4 \\ 4-3-12 & -4+12-\cancel{24}\,18 & 6+3-24 \end{bmatrix} = \begin{bmatrix} 12 & -2 & 20 \\ 13 & -23 & 8 \\ -11 & -10 & -15 \end{bmatrix}$

3. $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(a)

	diagonal	unit	symmetric	anti-symmetric
X	×	×	✓	×
Y	×	×	×	✓
Z	✓	×	✓	×

(b) $X^2 = \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (unit matrix), $X^2 = Y^2 = Z^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. $\underline{\underline{A}} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix}$, $\underline{\underline{B}} = \begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$, $\underline{\underline{C}} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$

The product of an $n_1 \times m_1$ matrix and a $n_2 \times m_2$ matrix is only defined iff $m_1 = n_2$, (giving a $n_1 \times m_2$ matrix).

(a) AB $(2,3) \times (2,3)$ not defined because $3 \neq 2$.

(b) $BC: (2,3) \times (3,2)$ is defined.

$$\begin{bmatrix} 4 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4+3+6 & 8+1+9 \\ 0+6+2 & 0+2+3 \end{bmatrix} = \begin{bmatrix} 13 & 18 \\ 8 & 5 \end{bmatrix}$$

(c) $$AB^T = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4+2+3 & 0+4+1 \\ 12+0+6 & 0+0+2 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 18 & 2 \end{bmatrix}$$

(d) AC^T is not defined: $(2,3) \times [(3,2)^T] = (2,3) \times (2,3)$ ✗.

(e) BC^T is not defined.

$$(BC)^T = \begin{pmatrix} 13 & 8 \\ 18 & 5 \end{pmatrix}$$

and $$C^T B^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4+3+6 & 0+6+2 \\ 8+1+9 & 0+2+3 \end{bmatrix} = \begin{pmatrix} 13 & 8 \\ 18 & 5 \end{pmatrix}$$ ✓

Week 5

$$\begin{matrix} 3 & -4 & 5 \\ 6 & -5 & -3 \\ -2 & 1 & 2 \end{matrix}$$

1. (a)

Minors of the elements of second row, denoted α_{2i}

$$\alpha_{21} = \begin{vmatrix} -4 & 5 \\ 1 & 2 \end{vmatrix} = -8-5 = -13, \quad \alpha_{22} = \begin{vmatrix} 3 & 5 \\ -2 & 2 \end{vmatrix} = 6+10 = 16$$

$$\alpha_{23} = \begin{vmatrix} 3 & -4 \\ -2 & 1 \end{vmatrix} = 3-8 = -5.$$

(b) Cofactors:

$$A_{21} = -\alpha_{21} = 13$$
$$A_{22} = +\alpha_{22} = 16$$
$$A_{23} = -\alpha_{23} = 5.$$

$$A_{ij} = (-1)^{i+j}\alpha_{ij}$$

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$= 6\times 13 + -5\times 16 + -3\times 5 = 78 - 80 - 15 = -17$$

(c) Now expand along first row,

$$\alpha_{11} = \begin{vmatrix} -5 & -3 \\ 1 & 2 \end{vmatrix} = -10+3 = -7 \quad \Rightarrow A_{11} = +\alpha_{11} = -7$$
$$\alpha_{12} = \begin{vmatrix} 6 & -3 \\ -2 & 2 \end{vmatrix} = 12 - 6 = 6 \quad \Rightarrow A_{12} = -\alpha_{12} = -6$$
$$\alpha_{13} = \begin{vmatrix} 6 & -5 \\ -2 & 1 \end{vmatrix} = 6-10 = -4 \quad \Rightarrow A_{13} = +\alpha_{13} = -4$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 3\times -7 + -4\times -6 + 5\times -4$$
$$= -21 + 24 - 20 = -17 \quad \checkmark$$

2. Alien Cofactor Rule: $a_{i1}A_{j1} + a_{i2}A_{j2} + a_{i3}A_{j3} = 0$ if $i \neq j$

$$a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 3\times 13 + -4\times 16 + 5\times 5$$
$$= 39 - 64 + 25 = 0 \quad \checkmark$$

$$a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23} = -2\times 13 + 1\times 16 + 2\times 5$$
$$= -26 + 16 + 10 = 0 \quad \checkmark$$

3.
$$\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = x\begin{vmatrix} x & a \\ a & x \end{vmatrix} - a\begin{vmatrix} a & a \\ a & x \end{vmatrix} + a\begin{vmatrix} a & x \\ a & a \end{vmatrix}$$
$$= x(x^2-a^2) - a(ax-a^2) + a(a^2-ax)$$
$$= x^3 - xa^2 - 2a^2x + 2a^3$$
$$= x^3 - 3a^2x + 2a^3$$
$$= (x-a)(x^2+ax-2a^2) = (x-a)(x-a)(x+2a)$$
$$= (x-a)^2(x+2a) \quad \checkmark$$

4. Can replace rows and columns with linear combinations to get

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{vmatrix} \underset{\substack{\text{equal to} \\ \text{determinant} \\ \text{of transpose}}}{=} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & c \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

5. $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} b & c \\ b^2 & c^2 \end{vmatrix} - \begin{vmatrix} a & c \\ a^2 & c^2 \end{vmatrix} + \begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix}$$

$$= bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b$$

$$= a^2(c-b) + a(b^2-c^2) + b(c-b)c$$

$$= [a^2 - a(b+c) + bc][c-b]$$

$$= (a-b)(a-c)(c-b)$$

$$= (a-b)(b-c)(c-a)$$

6. $A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \Rightarrow |A| = 2-(-3) = 5$

$B = \begin{bmatrix} -2 & 5 \\ 1 & -4 \end{bmatrix} \qquad |B| = 8-5 = 3$

$\Rightarrow |A||B| = 15$

$$AB = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -2 & 5 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} -5 & 10+4 \\ -6+1 & 15-4 \end{bmatrix} = \begin{bmatrix} -5 & 14 \\ -5 & 11 \end{bmatrix} \Rightarrow |AB| = -55+70 = 15$$

$$\Rightarrow |A||B| = |AB|$$

Week 6

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$

$A_{11} = \alpha_{11} = \begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} = 36-25 = 11$

$A_{12} = -\alpha_{12} = -\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -(12-5) = -7$

$A_{13} = \alpha_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 5-3 = 2$

$A_{21} = -\alpha_{21} = -\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} = -(24-15) = -9$

$A_{22} = \alpha_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 12-3 = 9$

$A_{23} = -\alpha_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -(5-2) = -3$

$A_{31} = \alpha_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10-9 = 1$

$A_{32} = -\alpha_{32} = -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -(5-3) = -2$

$A_{33} = \alpha_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3-2 = 1$

$$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$|A| = 1 \times 11 + -7 \times 2 + 3 \times 2 = 11-14+6 = 3$

$= 1 \times -9 + 3 \times 9 + 5 \times -3 = -9+27-15 = 3$ ✓

$$A^{-1} = \frac{1}{|A|}\text{adj}(A) = \frac{1}{3}\begin{pmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$A^{-1}A = \frac{1}{3}\begin{pmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 11-9+1 & 22-27+5 & 33-45+12 \\ -7+9-2 & -14+27-10 & -21+45-24 \\ 2-3+1 & 4-9+5 & 6-15+12 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$ ✓

2. $AB = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \Rightarrow (AB)^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix}$

$A^{-1} = \frac{1}{-3}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \quad B^{-1} = -\frac{1}{1}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}$

$\Rightarrow B^{-1}A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix} \quad \Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

3. $\underbrace{\begin{pmatrix} 4 & -3 & 1 \\ 2 & 1 & -4 \\ 1 & 2 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix}}_{B}$

$A_{11} = 6$
$A_{12} = 0$
$A_{13} = 3$

$$|A| = 4\begin{vmatrix} 1 & -4 \\ 2 & -2 \end{vmatrix} + 3\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} + 1\times\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 4\times(-2+8) + 3(-4+4) + 1\times(4-1) = 24+3 = 27$$

$A_{21} = -\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} = -(6-2) = -4,$ $\quad A_{31} = \begin{vmatrix} -3 & 1 \\ 1 & -4 \end{vmatrix} = 11$

$A_{22} = \begin{vmatrix} 4 & 1 \\ 1 & -2 \end{vmatrix} = -8-1 = -9,$ $\quad A_{32} = -\begin{vmatrix} 4 & 1 \\ 2 & -4 \end{vmatrix} = 18$

$A_{23} = -\begin{vmatrix} 4 & -3 \\ 1 & 2 \end{vmatrix} = -(8+3) = -11.$ $\quad A_{33} = \begin{vmatrix} 4 & -3 \\ 2 & 1 \end{vmatrix} = 10$

check: $|A| = 2\times -4 + 1\times -9 + -4\times -11 = -17+44 = 27$ ✓ (expanding on 2^{nd} row)

$|A| = 1\times 11 + 2\times 18 + -2\times 10 = 11+36-20 = 27$ ✓ 3^{rd} row.

$$\operatorname{adj}(A) = \begin{pmatrix} 6 & 0 & 3 \\ -4 & -9 & -11 \\ 11 & 18 & 10 \end{pmatrix}^T = \begin{pmatrix} 6 & -4 & 11 \\ 0 & -9 & 18 \\ 3 & -11 & 10 \end{pmatrix} \qquad A^{-1} = \frac{1}{|A|}\operatorname{adj}(A)$$

~~Altern~~ $X = A^{-1}B = \frac{1}{27}\begin{pmatrix} 6 & -4 & 11 \\ 0 & -9 & 18 \\ 3 & -11 & 10 \end{pmatrix}\begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 66+4+11 \\ 0+9+18 \\ 33+11+10 \end{pmatrix} = \frac{1}{27}\begin{pmatrix} 81 \\ 27 \\ 54 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

Week 7

1. $$\underbrace{\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & -\alpha \\ 1 & 2 & \alpha \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ Non-trivial solutions when matrix is singular, i.e. $|A| = 0$

$$|A| = 1\times\begin{vmatrix} 1 & -\alpha \\ 2 & \alpha \end{vmatrix} - 5\begin{vmatrix} 5 & -\alpha \\ 1 & \alpha \end{vmatrix} + 3\begin{vmatrix} 5 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= \alpha + 2\alpha - 5(5\alpha+\alpha) + 3\times(10-1)$$
$$= 3\alpha - 30\alpha + 27 = -27\alpha + 27$$

$|A| = 0 \Rightarrow \boxed{\alpha = 1}$.

Find solutions: let $x = \lambda$, so
$$\lambda + 5y + 3z = 0 \quad (1)$$
$$5\lambda + y - z = 0 \quad (2)$$

$(1) + 3(2) = \lambda + 15\lambda + 8y = 0 \Rightarrow y = -2\lambda$

$(1) - 5(2) = (1-25)\lambda + (3+5)z = 0 \Rightarrow 8z = 24\lambda \Rightarrow z = 3\lambda$

so solutions $\boxed{x = \lambda,\ y = -2\lambda,\ z = 3\lambda.}$

Check: $$\begin{pmatrix} 1 & 5 & 3 \\ 5 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1-10+9 \\ 5-2-3 \\ 1-4+3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$ ✓

2. $$\underbrace{\begin{bmatrix} \alpha+1 & -1 & 1-\alpha \\ 2 & 2-\alpha & -1 \\ 1 & 1 & -1 \end{bmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$ Non-trivial solutions for $|A| = 0$.

$$\begin{vmatrix} \alpha+1 & -1 & 1-\alpha \\ 2 & 2-\alpha & -1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} \alpha+2 & 0 & -\alpha \\ 1 & 1-\alpha & 0 \\ 1 & 1 & -1 \end{vmatrix} = (\alpha+2)\begin{vmatrix} 1-\alpha & 0 \\ 1 & -1 \end{vmatrix} - \alpha\begin{vmatrix} 1 & 1-\alpha \\ 1 & 1 \end{vmatrix}$$
$$= (\alpha+2)(\alpha-1) - \alpha - \alpha(\alpha-1) = \cancel{\alpha^2} + \alpha - 2 - \alpha - \cancel{\alpha^2} + \alpha$$
$$= \alpha - 2 \quad \Rightarrow |A| = 0 \quad \Rightarrow \boxed{\alpha = 2}$$

$$(\alpha+2)x - \alpha z = 0 \qquad \Rightarrow x = \lambda,\ z = \tfrac{1}{2}\times 4x = 2\lambda$$
$$x + (1-\alpha)y = 0 \qquad y = x = \lambda$$

so the solution is the line $\boxed{x = y = \lambda,\ z = 2\lambda.}$

3. $$\underbrace{\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}}_{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$ Is the matrix singular? Check determinant.

$$|A| = -2\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} - 1\times\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} + 1\times\begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} = -2\times(4-1) - (-2-1) + (1+2)$$
$$= -6 + 3 + 3 = 0 \Rightarrow \text{singular.}$$

The non-homogeneous singular system may have infinitely-many solutions, or it may be inconsistent.

Adding the three equations together,

$-2x + y + z + x - 2y + z + x + y - 2z = a + b + c$

$= 0$. So for the equations to be consistent, we require $a + b + c = 0$.

- Case $a = 1,\ b = 1,\ c = -2$.

Let $x = \lambda \Rightarrow -2\lambda + y + z = 1$ ①

$\lambda - 2y + z = 1$ ②

$2 \times ① + ② = -3\lambda + 3z = 3 \Rightarrow z = 1 + \lambda$

$① - ② = -3\lambda + 3y = 0 \Rightarrow y = \lambda$

ie. $\boxed{x = y = \lambda, \quad z = 1 + \lambda.}$

Check that this is consistent with final equation,

$x + y - 2z = \lambda + \lambda - 2(1+\lambda) = -2 = c$ ✓

4 (a) Gaussian elimination

Row	x	y	z	B	Sum	Opeation
(1)	4	−3	1	11	13	
(2)	2	1	−4	−1	−2	
* (3)	1	2	−2	1	2	
(4)	0	−11	9	7	5	(1) − 4(3)
(5)	0	−3	0	−3	−6	(2) − 2(3)

can now use back substitution: $-3y = -3 \Rightarrow y = 1$

$-11y + 9z = 7 \Rightarrow 9z = 18 \Rightarrow z = 2$

$x + 2y - 2z = 1 \Rightarrow x = 1 - 2 + 4 = 3$

Check: $4x - 3y + z = 12 - 3 + 2 = 11$ ✓

(b) Invert matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$ using Gaussian Elimination.

$$\begin{array}{c|ccc|ccc|l}
(1) & 1 & 2 & 3 & 1 & 0 & 0 & \\
(2) & 1 & 3 & 5 & 0 & 1 & 0 & \\
(3) & 1 & 5 & 12 & 0 & 0 & 1 & \\
\hline
 & 1 & 2 & 3 & 1 & 0 & 0 & \\
(4) & 0 & 1 & 2 & -1 & 1 & 0 & (2)-(1) \\
(5) & 0 & 3 & 9 & -1 & 0 & 1 & (3)-(1) \\
\hline
 & 1 & 2 & 3 & 1 & 0 & 0 & \\
 & 0 & 1 & 2 & -1 & 1 & 0 & \\
(6) & 0 & 0 & 3 & 2 & -3 & 1 & (5)-3(4)
\end{array}$$

$$\begin{array}{c|ccc|ccc|l}
 & 1 & 2 & 3 & 1 & 0 & 0 & \\
 & 0 & 1 & 2 & -1 & 1 & 0 & \\
(7) & 0 & 0 & 1 & 2/3 & -1 & 1/3 & (6)/3 \\
\hline
(8) & 1 & 2 & 0 & -1 & 3 & -1 & (1)-3(7) \\
(9) & 0 & 1 & 0 & -7/3 & 3 & -2/3 & (4)-2(7) \\
 & 0 & 0 & 1 & 2/3 & -1 & 1/3 & \\
\hline
 & 1 & 0 & 0 & 11/3 & -3 & 1/3 & \\
 & 0 & 1 & 0 & -7/3 & 3 & -2/3 & \\
 & 0 & 0 & 1 & 2/3 & -1 & 1/3 &
\end{array}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{11}{3} & -3 & \frac{1}{3} \\ -\frac{7}{3} & 3 & -\frac{2}{3} \\ \frac{2}{3} & -1 & \frac{1}{3} \end{bmatrix}$$

5. $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Eigenvalues are sol^{n}s of $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4$

$= (\lambda - 4)(\lambda + 1) = 0$

so $\lambda = 4$ and -1

$\lambda = 4$: $A - \lambda I = \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$ and $\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0 \Rightarrow x = 2, y = 3$

ie. eigenvector // to $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda = -1$: $A - \lambda I = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \Rightarrow$ eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$B = \begin{bmatrix} 4 & 0 & 1 \\ -1 & -6 & -2 \\ 5 & 0 & 0 \end{bmatrix}$. $|B - \lambda I| = \begin{vmatrix} 4-\lambda & 0 & 1 \\ -1 & -6-\lambda & -2 \\ 5 & 0 & -\lambda \end{vmatrix} = (4-\lambda)\begin{vmatrix} -6-\lambda & -2 \\ 0 & -\lambda \end{vmatrix} + 1\begin{vmatrix} -1 & -6-\lambda \\ 5 & 0 \end{vmatrix}$

$= (4-\lambda)(\lambda^2 + 6\lambda) + 5(6+\lambda)$

$= (\lambda+6)((4-\lambda)\lambda + 5)$

$= -(\lambda+6)(\lambda^2 - 4\lambda - 5)$

$= -(\lambda+6)(\lambda-5)(\lambda+1) = 0$

so eigenvalues -6, 5 and -1

$\lambda = -1$: $\begin{pmatrix} 5 & 0 & 1 \\ -1 & -5 & -2 \\ 5 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow z = -5x$

$-x - 5y - 2z = 9x - 5y = 0$

$5y = 9x$

$\Rightarrow \begin{pmatrix} 5 \\ 9 \\ -25 \end{pmatrix}$ eigenvector.

$\lambda = -6$: $\begin{pmatrix} 10 & 0 & 1 \\ -1 & 0 & -2 \\ 5 & 0 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 5$: $\begin{pmatrix} -1 & 0 & 1 \\ -1 & -11 & -2 \\ 5 & 0 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ y \\ 1 \end{pmatrix}$

where $-11y = 3 \Rightarrow y = -3/11$

So eigenvector $\begin{pmatrix} 11 \\ -3 \\ 11 \end{pmatrix}$

Week 7/8 Supplementary: Eigenvectors and Eigenvalues.

1. $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ (a) Characteristic eqn $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16 = 0$$
$$= \lambda^2 - 9 - 16 = 0$$

$\Rightarrow \lambda = -5, +5.$

(b) $(A - \lambda I)X = 0$. Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(i) $\lambda_1 = 5 \Rightarrow \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -2x_1 + 4x_2 = 0$
$x_1 = 2x_2$

Normalize: $\sqrt{x_1^2 + x_2^2} = 1$

$\Rightarrow \sqrt{4x_2^2 + x_2^2} = \sqrt{5}\,x_2 = 1 \Rightarrow x_2 = 1/\sqrt{5},\ x_1 = 2/\sqrt{5}.$

$\Rightarrow \hat{X}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (or $-\hat{X}_1$)

(ii) $\lambda_2 = -5 \Rightarrow \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x_2 = -2x_1$

$\Rightarrow \hat{X}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ (or $-\hat{X}_2$, also a valid normalized eigenvector).

(c) $X_1^T X_2 = \frac{1}{\sqrt{5}}(2\ \ 1)\begin{pmatrix} 1 \\ -2 \end{pmatrix} \times \frac{1}{\sqrt{5}} = \frac{1}{5}(2-2) = 0.$

(d) The eigenvectors of real symmetric matrices are orthogonal if $\lambda_1 \neq \lambda_2$. Proof: for $A = A^T$, consider

$$X_1^T(A - A^T)X_2 = 0$$
$$\Rightarrow X_1^T A X_2 - (AX_1)^T X_2 = \lambda_2 X_1^T X_2 - \lambda_1 X_1^T X_2 = 0$$
$$\Rightarrow (\lambda_2 - \lambda_1) X_1^T X_2 = 0$$ ie. either $X_1^T X_2 = 0$ or $\lambda_1 = \lambda_2$ (or both).

2. $A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$. $|A - \lambda I| = \begin{vmatrix} -\lambda & a \\ -a & -\lambda \end{vmatrix} = \lambda^2 + a^2 = 0$

$\Rightarrow \lambda = \pm ia.$

3x3 case: $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$. $|A - \lambda I| = \begin{vmatrix} -\lambda & a & b \\ -a & -\lambda & c \\ -b & -c & -\lambda \end{vmatrix}$

$$|A - \lambda I| = -\lambda\begin{vmatrix} -\lambda & c \\ -c & -\lambda \end{vmatrix} - a\begin{vmatrix} -a & c \\ -b & -\lambda \end{vmatrix} + b\begin{vmatrix} -a & -\lambda \\ -b & -c \end{vmatrix}$$
$$= -\lambda(\lambda^2 + c^2) - a(a\lambda + bc) + b(ac - \lambda b)$$
$$= -\lambda^3 - (a^2+b^2+c^2)\lambda - abc + bac = 0$$
$$= -\lambda(\lambda^2 + a^2 + b^2 + c^2) = 0$$

so one eigenvalue is zero, and the other two are

$$\lambda = \pm i\sqrt{a^2+b^2+c^2}$$

3(a) $\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = 0 \Rightarrow 1-\lambda = \pm 1$

$\Rightarrow \lambda = 1 \mp 1 = 0, 2$

(i) $\lambda = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X = \alpha\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(ii) $\lambda = 2 \Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X = \beta\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

where α, β are real numbers.

(b) $\begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = 0$

$= \lambda^2 - 3\lambda + 2 - 6 = \lambda^2 - 3\lambda - 4 = 0$

$= (\lambda - 4)(\lambda + 1) = 0 \Rightarrow \lambda = 4, -1$

(i) $\lambda = 4$: $\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow X = \alpha\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

(ii) $\lambda = -1$: $\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} X = 0 \Rightarrow X = \beta\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $\begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 5-\lambda & 4 \\ 4 & 3-\lambda \end{vmatrix} - 4\begin{vmatrix} 0 & 5-\lambda \\ -4 & 4 \end{vmatrix}$

$= (1-\lambda)((5-\lambda)(3-\lambda) - 16) + 4(5-\lambda)\times(-4)$

$= (1-\lambda)(\lambda^2 - 8\lambda - 1) - 16(5-\lambda)$

$= -\lambda^3 + 9\lambda^2 - 7\lambda - 1 + 16\lambda - 80 = -(\lambda^3 - 9\lambda^2 - 9\lambda + 81) = 0$

Try $\lambda = 3$: $27 - 81 - 27 + 81 = 0$ ✓

$\Rightarrow (\lambda - 3)(\lambda^2 - 6\lambda - 27) = (\lambda - 3)(\lambda - 9)(\lambda + 3)$

$\Rightarrow \lambda = -3, 3$ or 9.

(i) $\lambda = -3$: $\begin{bmatrix} 4 & 0 & -4 \\ 0 & 8 & 4 \\ -4 & 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x = z$

$y = -\frac{1}{2}z = -\frac{1}{2}x$

$\Rightarrow X = \alpha\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}$ (or multiples of this)

(ii) $\lambda = 3$: $\begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow x = y$

$2z = -y \Rightarrow z = -\frac{1}{2}y = -\frac{1}{2}x$

$\Rightarrow X = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \times \beta$

(iii) $\lambda = 9$: $\begin{bmatrix} -8 & 0 & -4 \\ 0 & -4 & 4 \\ -4 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow 2x = -z, \; y = z$

$\Rightarrow X = \gamma\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$

3.(d) $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix} \Rightarrow \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0 = (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 2 \\ -1 & 3-\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & 2-\lambda \\ -1 & 1 \end{vmatrix}$

$\Rightarrow (1-\lambda)((2-\lambda)(3-\lambda)-2) - 2 + 2(2-\lambda) = 0$

$(1-\lambda)(\lambda^2 - 5\lambda + 4) + 2(1-\lambda) = 0$

$\Rightarrow (1-\lambda)(\lambda^2 - 5\lambda + \overset{6}{\cancel{2}}) = 0$

$(1-\lambda)(\lambda-2)(\lambda-3) = 0 \Rightarrow \lambda = 1, 2, 3$

$\lambda = 1 : \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow y = -2z$, $x = y + 2z = 0 \Rightarrow X = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \times \alpha$

$\lambda = 2 : \begin{bmatrix} -1 & 1 & 2 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow z = 0,\ x = y \Rightarrow X = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \beta$

$\lambda = 3 : \begin{bmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow y = 2z$, $x = y \Rightarrow X = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \gamma$

Week 8

1 (a) $x^3\frac{dy}{dx} = 2x^2+3 \Rightarrow \frac{dy}{dx} = \frac{2x^2+3}{x^3} = \frac{2}{x} + \frac{3}{x^3}$

$\Rightarrow y = \int\left(\frac{2}{x} + \frac{3}{x^2}\right)dx = 2\ln(x) - \frac{3}{2}x^{-3} + c$

(b) $x(x+1)\frac{dy}{dx} = y-1$

$\Rightarrow \int\frac{dy}{y-1} = \int\frac{dx}{x(x+1)} \Rightarrow \ln|y-1| = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + c$

$= \ln|Ax/(x+1)|$ where $A = e^c$

$\Rightarrow y-1 = \frac{Ax}{x+1} \Rightarrow y = \frac{Ax}{x+1} + 1$

(c) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \Rightarrow \int\frac{dy}{1+y^2} = \int\frac{dx}{1+x^2} \Rightarrow \tan^{-1}(y) = \tan^{-1}(x) + c$

Use $\tan(a+b) = \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$ and let $c = \tan^{-1}(C)$

so $y = \tan(\tan^{-1}x + \tan^{-1}C) = \frac{x+C}{1-xC}$

(d) $(1+x)^2\frac{dy}{dx} + y^2 = 1$

$\Rightarrow \int\frac{dy}{1-y^2} = \int\frac{dx}{(1+x)^2} \Rightarrow \frac{1}{2}\int\left(\frac{1}{1-y} + \frac{1}{1+y}\right)dy = -\frac{1}{1+x} + c$

$\Rightarrow \frac{1}{2}\left(\ln|1+y| - \ln|1-y|\right) = -\frac{1}{1+x} + c$

$\Rightarrow \ln\left|\frac{1+y}{1-y}\right| = -\frac{2}{1+x} + c' \Rightarrow \frac{1+y}{1-y} = Ae^{-2/(1+x)}$ where $A = e^{c'}$

$\Rightarrow 1 - Ae^{-2/(1+x)} = -y\left(1 + Ae^{-2/(1+x)}\right)$

$\Rightarrow y = \frac{Ae^{-2/(1+x)} - 1}{Ae^{-2/(1+x)} + 1}$

(e) $(y-x)\frac{dy}{dx} = y$. Let $v = \frac{y}{x} \Rightarrow \frac{dv}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y$

$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + \frac{y}{x} = x\frac{dv}{dx} + v$

$\Rightarrow (vx-x)\left(x\frac{dv}{dx} + v\right) = vx$

$\Rightarrow x\frac{dv}{dx} + v = \frac{v}{v-1} \Rightarrow x\frac{dv}{dx} = v\left(\frac{1}{v-1} - 1\right) = \frac{v(2-v)}{v-1}$

$\Rightarrow \int\frac{v-1}{v(2-v)}dv = \int\frac{1}{x}dx = \ln(x) + c$

$= \frac{1}{2}\int\left(-\frac{1}{v} + \frac{1}{2-v}\right)dv = -\frac{1}{2}\left(\ln|v| + \ln|2-v|\right) = -\frac{1}{2}\ln|v(2-v)| = \ln|x| + c$

$\Rightarrow v(2-v) = Ax^{-2} \Rightarrow y(2x-y) = A$

$\Rightarrow y^2 - 2xy + A = 0 \Rightarrow (y-x)^2 - x^2 + A = 0$

$\Rightarrow (y-x)^2 = x^2 - A \Rightarrow y = x \pm \sqrt{x^2-A}$

Week 8 continued.

2. $x^3 \frac{dp}{dx} = a - x \Rightarrow \frac{dp}{dx} = \frac{a}{x^3} - \frac{1}{x^2} \Rightarrow p(x) = \frac{-a}{2x^2} + \frac{1}{x} + c$

$p(x=2) = 0 \Rightarrow 0 = -\frac{a}{8} + \frac{1}{2} + c$ ①

$p(x=4) = 0 \Rightarrow 0 = -\frac{a}{32} + \frac{1}{4} + c$ ②

Two eqns, two unknowns, a & c.

① − ② $= 0 = -\left(\frac{1}{8} - \frac{1}{32}\right)a + \frac{1}{2} - \frac{1}{4} \Rightarrow -\frac{3}{32}a + \frac{1}{4} = 0$

$\Rightarrow a = \frac{32}{4 \times 3} = 8/3$.

3. $x^2 \frac{dy}{dx} = y - \frac{dy}{dx}$

$\Rightarrow (x^2+1)\frac{dy}{dx} = y \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{1+x^2} \Rightarrow \ln|y| = \tan^{-1}(x) + c$

Try inserting $y=2$, $x = \tan(\ln 2)$

$\Rightarrow \ln(2) = \tan^{-1}(\tan(\ln 2)) + c = \ln(2) + c \Rightarrow c = 0$

hence, $\ln|y| = \tan^{-1}(x) \Rightarrow y = \exp(\tan^{-1}(x))$.

Week 9.

Integrating Factors (I.F.): If $\frac{dy}{dx} + P(x)\,y = Q(x)$, then the integrating factor is $F(x) = \exp\left(\int P(x)\,dx\right)$

1. $\frac{dy}{dx} + 3y = 5\sin(4x) \Rightarrow F(x) = \exp\left(\int 3\,dx\right) = e^{3x}$

$\Rightarrow e^{3x}\frac{dy}{dx} + 3y\,e^{3x} = 5\sin(4x)\,e^{3x}$

$\Rightarrow \frac{d}{dx}\left(e^{3x}y\right) = 5\sin(4x)\,e^{3x}$

$\Rightarrow e^{3x}y = \int 5\sin(4x)\,e^{3x}\,dx.$

$\Rightarrow e^{3x}y = \frac{1}{5}\left(3\sin 4x - 4\cos(4x)\right)e^{3x} + c$

$\Rightarrow y = \frac{1}{5}\left(3\sin 4x - 4\cos 3x\right) + ce^{-3x}$

Integration By Parts: Let $I = \int \underbrace{\sin(4x)}_{u}\,\underbrace{e^{3x}}_{v'}\,dx$

$\Rightarrow I = \sin(4x)\,\frac{1}{3}e^{3x} - \frac{4}{3}\int \cos(4x)\,e^{3x}\,dx$

$= \frac{1}{3}\sin(4x)\,e^{3x} - \frac{4}{9}\cos(4x)\,e^{3x} - \frac{16}{9}\int \sin(4x)\,e^{3x}\,dx$

$\Rightarrow \frac{25}{9}I = \left[\frac{1}{3}\sin(4x) - \frac{4}{9}\cos(4x)\right]e^{3x}$

$I = \frac{1}{25}\left[3\sin(4x) - 4\cos(4x)\right]e^{3x}$

2. $\frac{dy}{dx} + y = e^{x} \Rightarrow$ I.F. e^{x}

$e^{x}\frac{dy}{dx} + e^{x}y = e^{2x} \Rightarrow \frac{d}{dx}\left(e^{x}y\right) = e^{2x}$

$\Rightarrow e^{x}y = \frac{1}{2}e^{2x} + c$

$\Rightarrow y = \frac{1}{2}e^{x} + ce^{-x}$

3. $\frac{dy}{dx} + \frac{3y}{x} = x^3$. I.F. $\exp\left(\int \frac{3}{x}\,dx\right) = \exp\left(3\ln|x|\right) = x^3$

$\Rightarrow \frac{d}{dx}\left(x^3y\right) = x^6 \Rightarrow x^3y = \frac{1}{7}x^7 + c$

$\Rightarrow y = \frac{1}{7}x^4 + cx^{-3}$

4. $\frac{dy}{d\theta} + y\cot\theta = \sin\theta$. I.F. $\exp\left(\int \cot\theta\,d\theta\right) = \exp\left(\int \frac{\cos\theta}{\sin\theta}\,d\theta\right)$

$= \exp\left(\ln|\sin\theta|\right) = \sin\theta$

$\Rightarrow \frac{d}{d\theta}\left(\sin\theta\,y\right) = \sin^2\theta$

$= \frac{1}{2}\left(1 - \cos(2\theta)\right)$

$\Rightarrow \sin\theta \times y = \theta/2 - \frac{1}{4}\sin(2\theta) + c = \theta/2 - \frac{1}{2}\sin\theta\cos\theta + c$

$\Rightarrow y = \theta/(2\sin\theta) - \frac{1}{2}\cos\theta + \frac{c}{\sin\theta}$

Week 9 continued.

5. $(1-x^2)\frac{dy}{dx} - 1 = xy \Rightarrow (1-x^2)\frac{dy}{dx} - xy = 1$

$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{1-x^2}\right)y = \frac{1}{1-x^2}$

I.F. $\exp\left(\int \frac{-x}{1-x^2}\,dx\right) = \exp\left(\frac{1}{2}\ln(1-x^2)\right) = \sqrt{1-x^2}$

$\Rightarrow \frac{d}{dx}\left(\sqrt{1-x^2}\,y\right) = (1-x^2)^{-1/2}$

$\Rightarrow (1-x^2)^{1/2}\,y = \int\frac{dx}{\sqrt{1-x^2}} = \sin^{-1}(x) + c$

$\Rightarrow y = \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} + \frac{c}{\sqrt{1-x^2}}$.

6. $\frac{dy}{dx} + \frac{4xy}{x^2+1} = \frac{1}{(x^2+1)^2}$. IF. $\exp\left(\int\frac{4x}{x^2+1}\,dx\right) = \exp\left(2\ln|x^2+1|\right) = (x^2+1)^2$

$\Rightarrow \frac{d}{dx}\left((x^2+1)^2 y\right) = 1 \Rightarrow y = \frac{x+c}{(x^2+1)^2}$

7. $\frac{x}{y}\frac{dy}{dx} + 1 = xy$.

Let $v = 1/y \Rightarrow \frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -y^2\frac{dv}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$

$\Rightarrow -\frac{x}{v}\frac{dv}{dx} + 1 = x/v \Rightarrow \frac{dv}{dx} - \frac{v}{x} = -1.$

IF. $\exp\left(\int -\frac{dx}{x}\right) = \exp(-\ln x) = \frac{1}{x}$

$\Rightarrow \frac{d}{dx}\left(\frac{1}{x}v\right) = -1/x \Rightarrow v/x = -\ln(x) + c$

$\Rightarrow \frac{1}{y} = -x\ln(x) + cx \Rightarrow y = \frac{1}{x(c-\ln(x))}$

8. $\frac{dz}{dx} + 3\frac{z}{x} = 2z^2$. Following previous question, let $z = 1/v$

$\Rightarrow -\frac{1}{v^2}\frac{dv}{dx} + \frac{3}{xv} = \frac{2}{v^2} \Rightarrow \frac{dv}{dx} - \frac{3}{x}v = -2$

I.F. $\exp\left(\int -\frac{3}{x}\,dx\right) = x^{-3}$

$\Rightarrow vx^{-3} = \int(-2x^{-3})\,dx = x^{-2} + c$

$\Rightarrow \frac{1}{z} = x + cx^3 \Rightarrow z = \frac{1}{x(1+cx^2)}$

Week 10

1. (a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0.$ Try $y = e^{mx} \Rightarrow y' = me^{mx} = my$
$y'' = m^2e^{mx} = m^2y$

$\Rightarrow$ Auxiliary Equation : $m^2 + m - 6 = 0$
$\Rightarrow (m+3)(m-2) = 0 \quad \Rightarrow \quad m = 2, -3$

Solution: $y = Ae^{2x} + Be^{-3x}$ where A & B are arbitrary constants.

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 10y = 0 \quad \Rightarrow m^2 - 2m + 10 = 0$
$(m-1)^2 + 9 = 0$

$\Rightarrow (m-1)^2 = -9$
$\Rightarrow m - 1 = \pm 3i \quad \Rightarrow m = 1 \pm 3i.$ So roots are a complex-conjugate pair.

Solution: $y = e^{x}(A\cos(3x) + B\sin(3x))$

real part

imaginary part of root.

(c) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

$\Rightarrow$ Aux. Eq. $m^2 + 6m + 9 = 0 \quad \Rightarrow (m+3)^2 = 0 \Rightarrow$ repeated root at $x = -3.$

$\Rightarrow$ solution is $y = (A + Bx)e^{-3x}$

(d) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0 \quad \Rightarrow m^2 + 4m + 5 = 0$
$\Rightarrow (m+2)^2 + 1 = 0$

$\Rightarrow m = -2 \pm i$
$\Rightarrow y = e^{-2x}(A\cos x + B\sin x)$: General solution

Initial conditions: $y(0) = 0$ and $y'(0) = 2$
$\Rightarrow A = 0$ and $-2A + B = 2 \quad \Rightarrow B = 2,$ so
$y = 2\sin(x)\,e^{-2x}.$

2. (a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

The Particular Integral y_p is a solution with a similar form to the RHS.
Try $y_p = ae^{5x}$, where "a" is a constant to be determined.
Then $y_p' = 5y_p$, $y_p'' = 25y_p$
$\Rightarrow (25 - 3\times 5 + 2)\,ae^{5x} = e^{5x}$
$\Rightarrow 12a = 1 \qquad \Rightarrow a = 1/12 \Rightarrow y_p = \frac{1}{12}e^{5x}$

2(b) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = \sin(x)$

Try $y_p = a\cos(x) + b\sin(x)$

$y_p' = b\cos(x) - a\sin(x)$

$y_p'' = -a\cos(x) - b\sin(x)$

$\Rightarrow (-a+4b+5a)\cos(x) + (-b-4a+5b)\sin(x) = \sin(x)$

$\Rightarrow \quad -a+4b+5a = 4a+4b = 0 \quad ①$

$\quad\;\; -b-4a+5b = -4a+4b = 1 \quad ②$

① + ② $\Rightarrow 1 = 8b \Rightarrow b = 1/8,\ a = -1/8$, so

$y_p = \frac{1}{8}(\sin(x) - \cos(x))$.

(c) $\dfrac{d^2y}{dx^2} - y = 8xe^x$

Try $y_p = axe^x \Rightarrow y_p' = (ax+a)e^x,\ y_p'' = (ax+2a)e^x$

$\Rightarrow (ax+2a)e^x - axe^x = 2ae^x \neq 8xe^x$.

So the initial guess was not suitable.

Try instead $y_p = (bx^2+ax)e^x$

$y_p'' = (bx^2 + 4bx + 2b + ax + 2a)e^x$

$\Rightarrow y_p'' - y_p = (4bx + 2b + 2a)e^x = 8xe^x$

so $4b = 8 \Rightarrow b = 2$,

and $2b + 2a = 0 \qquad a = -b = -2$

$\Rightarrow y_p = (2x^2 - 2x)e^x = 2x(x-1)e^x$.

3. (a) $y'' - 3y' + 2y = e^{2x}$. First, find complementary function y_c, then particular integral y_p.

Aux. Eq.: $m^2 - 3m + 2 = 0$

$(m-2)(m-1) = 0 \quad \Rightarrow y_c = Ae^{2x} + Be^x$.

Note that the complementary function has a term "e^{2x}" of the same form as the term on the RHS of the equation. So we should modify our guess for the particular integral, (by raising the power of x by one).

$y_p = axe^{2x}$

$y_p' = (2ax + a)e^{2x}$

$y_p'' = (4ax + 4a)e^{2x} \quad \Rightarrow$ sub into differential equation,

$\Rightarrow [(4ax+4a) - 3(2ax+a) + 2ax]e^{2x} = e^{2x}$

$\Rightarrow ae^{2x} = e^{2x} \quad \Rightarrow a = 1 \quad \Rightarrow y_p = xe^{2x}$.

$\Rightarrow$ General Solution: $y = xe^{2x} + Ae^{2x} + Be^x$.

3(b) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = xe^{2x}$

CF. $m^2 + m + 1 = 0 \Rightarrow (m + 1/2)^2 = -3/4$

$m = -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$.

$y_c = e^{-\frac{1}{2}x}\left(A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$.

PI. Try $y_p = (ax+b)e^{2x}$

$y_p' = (2ax + a + 2b)e^{2x}$

$y_p'' = (4ax + 4a + 4b)e^{2x}$

$\Rightarrow (4ax + 4a + 4b + 2ax + a + 2b + ax + b)e^{2x} = x\, e^{2x}$

$\Rightarrow 7ax = x \qquad \Rightarrow a = 1/7$

and $5a + 7b = 0 \qquad \Rightarrow b = -5/49 \qquad \Rightarrow y_p = \left(\frac{1}{7}x - \frac{5}{49}\right)e^{2x}$

$\Rightarrow$ General Solution: $y = \left(\frac{1}{7}x - \frac{5}{49}\right)e^{2x} + e^{-x/2}\left(A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$.

3(c) $y'' - y = e^x\sin(x)$.

CF. $m^2 - 1 = 0 \Rightarrow m = \pm 1 \Rightarrow y_c = Ae^x + Be^{-x}$

PI: Try $y = [a\cos(x) + b\sin(x)]e^x$

$y' = [(a+b)\cos(x) + (b-a)\sin(x)]e^x$

$y'' = [2b\cos(x) - 2a\sin(x)]e^x$

$\Rightarrow [(2b-a)\cos(x) - (b+2a)\sin(x)]\cancel{e^x} = \cancel{e^x}\sin(x)$

$\Rightarrow \quad 2a + b = -1 \qquad \Rightarrow 5b = -1 \Rightarrow b = -1/5,\ a = -2/5$

$-a + 2b = 0 \Rightarrow a = 2b$

$\Rightarrow y = -\left(\frac{2}{5}\cos(x) + \frac{1}{5}\sin(x)\right)e^x + Ae^x + Be^{-x}$.

(d) $y'' - 2y' + y = (3x^2 + 2x)e^x + 5x^2 - 6x$

CF. $m^2 - 2m + 1 = (m-1)^2 = 0 \Rightarrow$ repeated root, $m = 1$

$\Rightarrow y_c = (A + Bx)e^x$.

PI. ~~Try~~ $\cancel{y_p = (ax^3 + bx)e^x + cx^2 + dx + e}$

$\cancel{y_p' = (ax^2 + (2a+b)x + b)e^x + 2cx + d}$

$\cancel{y_p'' = (ax^2 + (4a+b)x + 2a + 2b)e^x + 2c}$

Split P.I. into two bits: $y_p = y_1 + y_2$

y_1 takes care of polynomial, this is "easy" part, try

$y_1 = ax^2 + bx + c \qquad \Rightarrow 2a - 2b + c = 0$

$y_1' = 2ax + b \qquad -4a + b = -6$

$y_1'' = 2a \qquad a = 5$

$\Rightarrow b = -6 + 20 = 14, \quad c = 2b - 2a = 28 - 10 = 18$

$y_1 = 5x^2 + 14x + 18$.

Now try $y_2 = (\alpha x^4 + \beta x^3)e^x$,

raising power by two because of clash with double root in CF.

$$y_2' = (\alpha x^4 + (4\alpha+\beta)x^3 + 3\beta x^2)e^x$$

$$y_2'' = (\alpha x^4 + (8\alpha+\beta)x^3 + (12\alpha+6\beta)x^2 + 6\beta x)e^x$$

Check: $\alpha - 2\alpha + \alpha = 0$ ✓ (x^4 coefficient)

and $(8\alpha+\beta) - 2(4\alpha+\beta) + \beta = 0$ ✓ (x^3 coefficient),

leaving $\cancel{-2\times 3\beta} + 12\alpha \cancel{+6\beta} = 3$ (x^2 coefficient),

and $6\beta = 2$ (x coefficient)

$\Rightarrow \beta = 1/3$, $\Rightarrow \alpha = 1/4$.

General Solution: $y = (A + Bx + \frac{1}{3}x^3 + \frac{1}{4}x^4)e^x + 5x^2 + 14x + 18$.

Week 11

1. $A = \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$. Characteristic Eq $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = (3-\lambda)(-3-\lambda) - 16 = \lambda^2 - 9 - 16 = \lambda^2 - 25 = 0$$

$\Rightarrow \lambda = \pm 5.$

Case $\lambda_1 = 5$: $\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} 2x = 4y \\ x = 2y \end{matrix} \Rightarrow X_1 = \alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Case $\lambda_2 = -5$: $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 8x + 4y = 0 \Rightarrow 2x = -y \Rightarrow X_2 = \beta\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Hence the solution to $\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = e^{\lambda_1 t} X_1 + e^{\lambda_2 t} X_2 = \begin{pmatrix} 2\alpha e^{5t} + \beta e^{-5t} \\ \alpha e^{5t} - 2\beta e^{-5t} \end{pmatrix}$$

ie. $x_1(t) = 2\alpha e^{5t} + \beta e^{-5t}$,
$x_2(t) = \alpha e^{5t} - 2\beta e^{-5t}$, where α & β are arbitrary constants.

Now apply the initial conditions $x_1(0) = 1$ and $x_2(0) = 3$

$\Rightarrow \begin{matrix} 2\alpha + \beta = 1 \\ \alpha - 2\beta = 3 \end{matrix} \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$\Rightarrow -\frac{1}{5}\begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}$ is the inverse, so

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2+3 \\ 1-6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{so}$$

ie. $x_1(t) = 2e^{5t} - e^{-5t}$
$x_2(t) = e^{5t} + 2e^{-5t}$.

2. ~~expanding into~~

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = -(1)\times\begin{vmatrix} 1-\lambda & -2 \\ -1 & 1 \end{vmatrix} - (1+\lambda)\begin{vmatrix} 1-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix}$$ (expanding along bottom row)

$$= (\lambda - 1 + 2) - (1+\lambda)\big((1-\lambda)(2-\lambda) + 1\big)$$
$$= (\lambda+1) - (\lambda+1)\big((1-\lambda)(2-\lambda) + 1\big)$$
$$= -(\lambda+1)(1-\lambda)(2-\lambda) = 0$$

Eigenvectors. Case $\lambda = 1$: $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} y = 2z \\ X = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \end{matrix}$

Case $\lambda = -1$: $\begin{bmatrix} 2 & 1 & -2 \\ -1 & 3 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow X = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Week 11 (continued).

2. continued. Case $\lambda = 2$: $\begin{bmatrix} -1 & 1 & -2 \\ -1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow X = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

$\Rightarrow$ General Solution: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} e^{t} \alpha + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-t} \beta + \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} e^{2t} \gamma,$

$$x_1 = 3Ae^{t} + Be^{-t} + Ce^{2t},$$
$$x_2 = 2Ae^{t} + 3Ce^{2t},$$
$$x_3 = Ae^{t} + Be^{-t} + Ce^{2t}.$$

$\alpha = A,\ \beta = B,\ \gamma = C,$