Examples 7: Vectors

1. If $\mathbf{a}$ and $\mathbf{b}$ are vectors as given in the table below, verify that the scalar products are as shown. Which of the three pairs of vectors is perpendicular?

	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{a}.\mathbf{b}$
(i)	$(1,-1,0)$	$(3,4,5)$	-1
(ii)	$(4,1,-3)$	$(-1,3,-7)$	20
(iii)	$(3,1,4)$	$(2,-2,-1)$	0

2. If $\mathbf{a} = (0,-1,1)$ and $\mathbf{b} = (3,4,5)$, show that

$$a = |\mathbf{a}| = \sqrt{2} \quad \text{and} \quad b = |\mathbf{b}| = 5\sqrt{2}.$$

Hence, using the formula $\mathbf{a}.\mathbf{b} = ab\cos\theta$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

3. If $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ prove the unit vector parallel to $\mathbf{a}$ is

$$\hat{\mathbf{a}} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{1}{\sqrt{6}}\mathbf{j} + \frac{2}{\sqrt{6}}\mathbf{k}.$$

4. Show that the vectors $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}-2\lambda\mathbf{j}+\mathbf{k}$ are perpendicular if (and only if) $\lambda = 1$.

5. Find the constants λ and μ, given that the vector $(\lambda, 4, \mu)$ is perpendicular to each of the vectors $(1,2,3)$ and $(-1,1,1)$.

6. If $\mathbf{a} = (2,5,-2)$, $\mathbf{b} = (4,-1,-2)$, $\mathbf{c} = (0,2,0)$ and $\mathbf{d} = (8,10,-6)$, express the vector $\mathbf{d}$ as the sum of three vectors, one parallel to the vector $\mathbf{a}$, one parallel to the vector $\mathbf{b}$ and one parallel to the vector $\mathbf{c}$.

7. Prove (using vectors) that the perpendiculars onto the sides of a triangle are concurrent (i.e. meet at a point). [Hint: draw the perpendiculars from the vertices A, B of a triangle ABC and let them intersect at O. Let the position vectors of A, B, C relative to O be $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$. Show that

$$\mathbf{a}.(\mathbf{c}-\mathbf{b}) = 0 = \mathbf{b}.(\mathbf{c}-\mathbf{a}).$$

Deduce that $\mathbf{c}.(\mathbf{b}-\mathbf{a}) = 0$ and interpret this result.]

8. Show that, if the vectors $\mathbf{a}$ and $\mathbf{b}$ are as given in the first two columns of the table below, $\mathbf{a}\times\mathbf{b}$ is as given in the third column. Show also that the vectors in the fourth column are unit vectors perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

	$\mathbf{a}$	$\mathbf{b}$	$\mathbf{a}\times\mathbf{b}$	
(i)	$(3,7,2)$	$(1,3,1)$	$(1,-1,2)$	$\pm\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$
(ii)	$(1,-3,0)$	$(-2,5,0)$	$(0,0,-1)$	$(0,0,\pm1)$
(iii)	$(8,8,-1)$	$(5,5,2)$	$(21,-21,0)$	$\pm\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

9. Find a and b if $(a\mathbf{i} + b\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = \mathbf{i} - \mathbf{j}$.

10. If the vectors $(3, -4, \lambda)$ and $(2, b, 3)$ are parallel, find λ and b.

11. If $\mathbf{a} = (2, 5, -2)$, $\mathbf{b} = (4, -1, -2)$ and $\mathbf{c} = (1, -2, -1)$
    (a) Find $(\mathbf{a} \times \mathbf{b}).\mathbf{c}$, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$, $(\mathbf{a}.\mathbf{b})\mathbf{c} + \dfrac{12\mathbf{b}}{\mathbf{a}.\mathbf{c}}$.
    (b) Verify, for these particular vectors, the identity $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a}$.

12. Show that $\mathbf{v}.(\mathbf{v} \times \mathbf{B}) = 0$. Deduce that if $\dfrac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{B}$, then v is constant.

13. The position vector $\mathbf{r}$ of a particle at time t is given by $\mathbf{r} = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j}$, where a, b and ω are positive constants with $a > b$. Find the particle's velocity $\mathbf{v}$ and show that its speed is given by

$$\omega\sqrt{b^2 + (a^2 - b^2)\sin^2\omega t}.$$

Find also the acceleration, $\mathbf{f}$, of the particle and the time when $\mathbf{f}$ is perpendicular to $\mathbf{v}$.

14. If $\mathbf{r} = a(\sin\omega t, \cos\omega t, 0)$ and

$$m\frac{d^2\mathbf{r}}{dt^2} = eB\frac{d\mathbf{r}}{dt} \times \mathbf{k},$$

(where a, ω, m, e and B are constants) prove that $\omega = eB/m$.

Answers

1. (iii)
2. $\cos^{-1}(1/10)$
5. $\lambda = 1$, $\mu = -3$
6. $\mathbf{d} = 2(2, 5, -2) + (4, -1, -2) + \frac{1}{2}(0, 2, 0)$
9. $a = 1 = b$
10. $\lambda = 9/2$, $b = -8/3$
11. 18, $(-40, -34, 28)$, $(-1, -12, -3)$
13. $\mathbf{v} = -\omega a\sin\omega t\mathbf{i} + \omega b\cos\omega t\mathbf{j}$, $\mathbf{f} = -\omega^2\mathbf{r}$, $\mathbf{f}.\mathbf{v} = 0$ when $t = \dfrac{n\pi}{2\omega}$, $(n = 0, \pm1, \pm2, \ldots)$