Solutions to Vectors

1. (*i*) $(1,-1,0).(3,4,5) = 1\times 3+(-1)\times 4+0\times 5 = -1$
   (*ii*) $(4,1,-3).(-1,3,-7) = 4\times(-1)+1\times 3+(-3)\times(-7) = 20$
   (*iii*) $(3,1,4).(2,-2,-1) = 3\times 2+1\times(-2)+4\times(-1) = 0 \Rightarrow$ **a** and **b** are perpendicular.

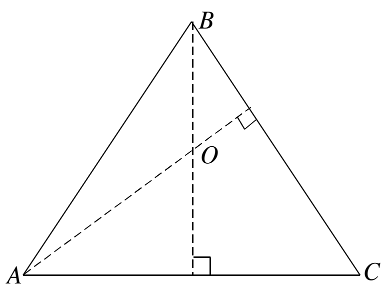
2. $\mathbf{a} = (0,-1,1) \Rightarrow a = |\mathbf{a}| = \sqrt{0^2+(-1)^2+1^2} = \sqrt{2}$
   $\mathbf{b} = (3,4,5) \Rightarrow b = |\mathbf{b}| = \sqrt{3^2+4^2+5^2} = \sqrt{50} = 5\sqrt{2}$
   $\cos\theta = \dfrac{\mathbf{a}.\mathbf{b}}{ab} = \dfrac{(0,-1,1).(3,4,5)}{\sqrt{2}\times 5\sqrt{2}} = \dfrac{0-4+5}{10} = 0.1 \Rightarrow \theta = \cos^{-1}0.1 \approx 1.47$

3. $\mathbf{a} = \mathbf{i}+\mathbf{j}+2\mathbf{k} = (1,1,2) \Rightarrow a = |\mathbf{a}| = \sqrt{1+1+4} = \sqrt{6}$
   and $\hat{\mathbf{a}} = \dfrac{\mathbf{a}}{a} = \dfrac{1}{\sqrt{6}}(\mathbf{i}+\mathbf{j}+2\mathbf{k}) = \dfrac{1}{\sqrt{6}}\mathbf{i}+\dfrac{1}{\sqrt{6}}\mathbf{j}+\dfrac{2}{\sqrt{6}}\mathbf{k}$

4. $(\mathbf{i}+\mathbf{j}+\mathbf{k}).(\mathbf{i}-2\lambda\mathbf{j}+\mathbf{k}) = (1,1,1).(1,-2\lambda,1) = 1-2\lambda+1 = 2(1-\lambda) = 0$ when (and only when) $\lambda = 1$.

5. If $\mathbf{r} = (\lambda,4,\mu)$ is perpendicular to $\mathbf{a} = (1,2,3)$ and to $\mathbf{b} = (-1,1,1)$ then $\mathbf{r}.\mathbf{a} = 0$ and $\mathbf{r}.\mathbf{b} = 0$.
   But $\mathbf{r}.\mathbf{a} = \lambda+8+3\mu$ and $\mathbf{r}.\mathbf{b} = -\lambda+4+\mu$ and so
   $$\left.\begin{array}{r} \lambda+3\mu=-8 \\ -\lambda+\mu=-4 \end{array}\right\} \Rightarrow \left.\begin{array}{r} \lambda=1 \\ \mu=-3 \end{array}\right\}$$

6. Suppose that $\mathbf{d} = \alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c}$. Now $\mathbf{b}\times\mathbf{c}$ is perpendicular to both **b** and **c**, so $(\mathbf{b}\times\mathbf{c}).\mathbf{b} = 0$ and $(\mathbf{b}\times\mathbf{c}).\mathbf{c} = 0$. Hence
   $$\mathbf{d}.(\mathbf{b}\times\mathbf{c}) = \alpha\mathbf{a}.(\mathbf{b}\times\mathbf{c}) \Rightarrow \alpha = \frac{\mathbf{d}.(\mathbf{b}\times\mathbf{c})}{\mathbf{a}.(\mathbf{b}\times\mathbf{c})}$$
   But $\mathbf{b}\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & -1 & -2 \\ 0 & 2 & 0 \end{vmatrix} = (4,0,8) \Rightarrow \alpha = \dfrac{(8,10,-6).(4,0,8)}{(2,5,-2).(4,0,8)} = \dfrac{-16}{-8} = 2$
   Likewise $\beta = \dfrac{\mathbf{d}.(\mathbf{a}\times\mathbf{c})}{\mathbf{b}.(\mathbf{a}\times\mathbf{c})} = \dfrac{(8,10,-6).(4,0,4)}{(4,-1,-2).(4,0,4)} = \dfrac{8}{8} = 1$ and
   $\gamma = \dfrac{\mathbf{d}.(\mathbf{a}\times\mathbf{b})}{\mathbf{c}.(\mathbf{a}\times\mathbf{b})} = \dfrac{(8,10,-6).(-12,-4,-22)}{(0,2,0).(-12,-4,-22)} = \dfrac{-4}{-8} = \dfrac{1}{2}$
   Finally $\mathbf{d} = \alpha\mathbf{a}+\beta\mathbf{b}+\gamma\mathbf{c} = 2\mathbf{a}+\mathbf{b}+\frac{1}{2}\mathbf{c}$

7. 

   Since OA is perpendicular to BC and $\overrightarrow{BC} = \mathbf{c}-\mathbf{b}$, we have $\mathbf{a}.(\mathbf{c}-\mathbf{b}) = 0$. Likewise OB being perpendicular to AC implies that $\mathbf{b}.(\mathbf{c}-\mathbf{a}) = 0$.
   Hence $\mathbf{a}.\mathbf{b} = \mathbf{a}.\mathbf{c} = \mathbf{b}.\mathbf{c} \Rightarrow \mathbf{c}.(\mathbf{a}-\mathbf{b}) = 0$ and so OC is perpendicular to AB. That is, O lies on all three of the perpendiculars.

8. (i) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 7 & 2 \\ 1 & 3 & 1 \end{vmatrix} = (7\times 1 - 2\times 3, 2\times 1 - 3\times 1, 3\times 3 - 7\times 1) = (1,-1,2)$ which has magnitude $\sqrt{1+1+4} = \sqrt{6}$ and so the required unit vector is $\dfrac{\pm 1}{\sqrt{6}}(1,-1,2)$

(ii) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 0 \\ -2 & 5 & 0 \end{vmatrix} = (0,0,1\times 5 - (-3)\times(-2)) = (0,0,-1)$

(iii) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 8 & -1 \\ 5 & 5 & 2 \end{vmatrix} = (8\times 2 - (-1)\times 5, (-1)\times 5 - 8\times 2, 8\times 5 - 8\times 5) = (21,-21,0)$

which has magnitude $21\sqrt{2}$ and so the required unit vector is $\dfrac{\pm 1}{\sqrt{2}}(1,-1,0)$

9. $(a\mathbf{i}+b\mathbf{j}+\mathbf{k})\times(2\mathbf{i}+2\mathbf{j}+3\mathbf{k}) = (a,b,1)\times(2,2,3) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & 1 \\ 2 & 2 & 3 \end{vmatrix} = (3b-2, 2-3a, 2a-2b) = \mathbf{i}-\mathbf{j}$

Hence $\left.\begin{aligned} 3b-2 &= 1 \\ 2-3a &= -1 \\ 2a-2b &= 0 \end{aligned}\right\}$ and all three of these equations are satisfied by $a=1$ and $b=1$.

10. $(3,-4,\lambda)$ and $(2,b,3)$ being parallel implies that one is a multiple of the other. Hence $(3,-4,\lambda) = \alpha(2,b,3)$ for some number α which gives $\left.\begin{aligned} 3 &= 2\alpha \\ -4 &= \alpha b \\ \lambda &= 3\alpha \end{aligned}\right\}$

$\Rightarrow \alpha = \frac{3}{2},\ b = -\frac{8}{3}$ and $\lambda = \frac{9}{2}$.

11. (i) $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & -2 \\ 4 & -1 & -2 \end{vmatrix} = (-10-2, -8+4, -2-20) = (-12,-4,-22)$

$\Rightarrow (\mathbf{a}\times\mathbf{b}).\mathbf{c} = (-12,-4,-22).(1,-2,-1) = -12+8+22 = 18$

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -12 & -4 & -22 \\ 1 & -2 & -1 \end{vmatrix} = (4-44, -22-12, 24+4) = (-40,-34,28)$

$\mathbf{a}.\mathbf{b} = (2,5,-2).(4,-1,-2) = 8-5+4 = 7$ and

$\mathbf{a}.\mathbf{c} = (2,5,-2).(1,-2,-1) = 2-10+2 = -6$

$(\mathbf{a}.\mathbf{b})\mathbf{c} + \dfrac{12\mathbf{b}}{\mathbf{a}.\mathbf{c}} = 7(1,-2,-1) + \dfrac{12}{(-6)}(4,-1,-2) = (-1,-12,-3)$

(ii) $\mathbf{b}.\mathbf{c} = (4,-1,-2).(1,-2,-1) = 4+2+2 = 8$

$(\mathbf{a}.\mathbf{c})\mathbf{b} - (\mathbf{b}.\mathbf{c})\mathbf{a} = -6(4,-1,-2) - 8(2,5,-2) = (-24-16, 6-40, 12+16)$
$= (-40,-34,28) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$, as required.

12. Because $(\mathbf{v}\times\mathbf{B})$ is perpendicular to $\mathbf{v}$, the dot product $(\mathbf{v}\times\mathbf{B}).\mathbf{v}$ is zero. Thus

$$\frac{d(v^2)}{dt} = \frac{d\mathbf{v}.\mathbf{v}}{dt} = 2\mathbf{v}.\frac{d\mathbf{v}}{dt} = 2\mathbf{v}.(\mathbf{v}\times\mathbf{B}) = 0$$

Hence $v = |\mathbf{v}|$ is a constant.

13. $\mathbf{r} = a\cos\omega t\mathbf{i} + b\sin\omega t\mathbf{j} \Rightarrow \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = -a\omega\sin\omega t\mathbf{i} + b\omega\cos\omega t\mathbf{j}$

$\Rightarrow v = |\mathbf{v}| = \sqrt{(a\omega\sin\omega t)^2 + (b\omega\cos\omega t)^2} = \omega\sqrt{a^2\sin^2\omega t + b^2(1-\sin^2\omega t)}$
$= \omega\sqrt{b^2 + (a^2-b^2)\sin^2\omega t}$

$\mathbf{f} = \dfrac{d\mathbf{v}}{dt} = -a\omega^2\cos\omega t\mathbf{i} - b\omega^2\sin\omega t\mathbf{j} = -\omega^2\mathbf{r}$

$\mathbf{f}.\mathbf{v} = -\omega^2(a\cos\omega t, b\sin\omega t, 0).(-a\omega\sin\omega t, b\omega\cos\omega t, 0) = \omega^3(a^2-b^2)\sin\omega t\cos\omega t$

which is zero when $\sin(2\omega t) = 0 \Rightarrow 2\omega t = n\pi$ (n any integer) and so $\mathbf{f}$ and $\mathbf{v}$ are perpendicular when $t = \dfrac{n\pi}{2\omega}$.

14. $\mathbf{r} = a(\sin\omega t, \cos\omega t, 0) \Rightarrow \dfrac{d\mathbf{r}}{dt} = a\omega(\cos\omega t, -\sin\omega t, 0)$ and $\dfrac{d^2\mathbf{r}}{dt^2} = -a\omega^2(\sin\omega t, \cos\omega t, 0)$

Also $\dfrac{d\mathbf{r}}{dt} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\omega\cos\omega t & -a\omega\sin\omega t & 0 \\ 0 & 0 & 1 \end{vmatrix} = -a\omega(\sin\omega t, \cos\omega t, 0)$.

Hence $m\dfrac{d^2\mathbf{r}}{dt^2} = eB\dfrac{d\mathbf{r}}{dt} \times \mathbf{k} \Rightarrow ma\omega^2 = eBa\omega \Rightarrow \omega = \dfrac{eB}{m}$.

[This describes the motion of an electron of charge e and mass m in a magnetic field of strength $\mathbf{B} = B\mathbf{k}$. ω is the gyrofrequeny, the rate at which the electron spirals around the magnetic field lines.]