Examples 6: Complex Numbers 2

1. Find the three roots of the equation $(z-1)^3 = 8i$ and plot them on the Argand diagram.

2. Use the result $2i\sin\theta = e^{i\theta} - e^{-i\theta}$ to establish the identity

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$

3. Show that $\dfrac{1+i\sqrt{3}}{1-i\sqrt{3}} = e^{2i\pi/3}$.
Hence express $\{(1+i\sqrt{3})/(1-i\sqrt{3})\}^{13}$ in the form $a+ib$, where a and b are real.

4. Express $1+i,\ 1+i\sqrt{3}$ and $\sqrt{3}-i$ in modulus-argument form and **hence** find the modulus and principal argument of the complex number

$$z = \frac{8(1+i)^6}{(1+i\sqrt{3})^2(\sqrt{3}-i)^3}.$$

5. (a) State de Moivre's theorem and use it to prove that

$$\left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n = \cos n\theta + i\sin n\theta.$$

   (b) If $z_1 = e^{i\pi/3}$ and $z_2 = e^{-i\pi/4}$ express the numbers ξ_1, ξ_2 and ξ_3 where

$$\xi_1 = z_1/z_2^2,\ \xi_2 = z_1^2 z_2^4 \text{ and } \xi_3 = 1/z_2^4$$

   in the form $a+ib$ (where a and b are real) and plot them on the Argand diagram.

6. By writing a complex number z in the form $z = re^{i\theta}$, and using the relation $\ln(ab) = \ln a + \ln b$, show that the natural logarithm of a general complex number can be defined as

$$\ln z = \ln|z| + i\arg z.$$

   Find all values of $\ln(-2-3i)$.

Answers

1. $(1+\sqrt{3})+i, (1-\sqrt{3})+i, 1-2i.$

3. $-1/2 + i\sqrt{3}/2.$

4. 2, $-2\pi/3$.

5. (b) $\xi_1 = (-\sqrt{3}+i)/2$, $\xi_2 = (1-i\sqrt{3})/2$, $\xi_3 = -1+i0$.

6. $\ln\sqrt{13} + (2n\pi - 2.16)i, \quad n \in \mathbb{N}.$