MA3140/151/152 EXAMPLE SHEET 6 SOLUTIONS

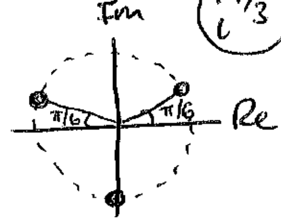
1) $(z-1)^3 = 8i \Rightarrow z-1 = (8i)^{1/3}$
$$= 8^{1/3}\, i^{1/3}$$
$$= 2\, i^{1/3}.$$

Write i as $e^{i\pi/2}$, $e^{i\pi/2 - i2\pi}$, $e^{i\pi/2 + i2\pi}$

(or any three arguments with 2π differences)

Then $i^{1/3} = e^{i\pi/6}, e^{i\pi/6 - i2\pi/3}, e^{i\pi/6 + i2\pi/3}$
$$= e^{i\pi/6}, e^{-i\pi/2}, e^{i5\pi/6}.$$

$i^{1/3}$



$$\therefore \quad z-1 = \begin{cases} 2e^{i\pi/6} \\ 2e^{-i\pi/2} \\ 2e^{i5\pi/6} \end{cases}$$

$$\therefore \quad z = \begin{cases} 1 + 2e^{i\pi/6} = (1+\sqrt{3}) + i \\ 1 + 2e^{-i\pi/2} = 1 - 2i \\ 1 + 2e^{i5\pi/6} = (1-\sqrt{3}) + i \end{cases}$$

since $\cos \pi/6 = \frac{\sqrt{3}}{2}$

$\sin \pi/6 = \frac{1}{2}$

$\cos 5\pi/6 = -\frac{\sqrt{3}}{2}$

$\sin 5\pi/6 = \frac{1}{2}$

2) $2i\sin\theta = e^{i\theta} - e^{-i\theta}$

$$\therefore \quad (2i\sin\theta)^4 = \left(e^{i\theta} - e^{-i\theta}\right)^4$$

$$2^4 i^4 \sin^4\theta = \left(e^{i\theta}\right)^4 + 4\left(e^{i\theta}\right)^3\left(-e^{-i\theta}\right) + 6\left(e^{i\theta}\right)^2\left(-e^{-i\theta}\right)^2 + 4\left(e^{i\theta}\right)\left(-e^{-i\theta}\right)^3 + \left(-e^{-i\theta}\right)^4$$

$$\therefore \quad 16\sin^4\theta = e^{i4\theta} - 4e^{i3\theta}e^{-i\theta} + 6e^{i2\theta}e^{-i2\theta} \cancel{+} - 4e^{i\theta}e^{-i3\theta} + e^{-i4\theta}$$

$$= \left(e^{i4\theta} + e^{-i4\theta}\right) - 4\left(e^{i2\theta} - e^{-i2\theta}\right) + 6$$

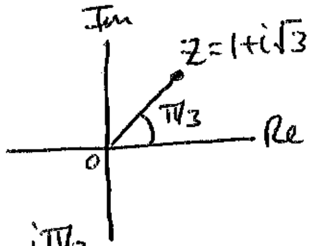
$$= 2\cos4\theta - 8\cos2\theta + 6$$

$$\therefore \quad \sin^4\theta = \frac{1}{8}\left(\cos4\theta - 4\cos2\theta + 3\right)$$

3) Let $z = 1 + i\sqrt{3}$

then $|z| = \sqrt{1+3} = \sqrt{4} = 2.$

$\text{Arg}\, z = \tan^{-1}\sqrt{3} = \pi/3.$ $\therefore \underline{\underline{z = 2e^{i\pi/3}}}$



Then $\bar{z} = 1 - i\sqrt{3} = 2e^{-i\pi/3}$

$$\therefore \frac{1+i\sqrt{3}}{1-i\sqrt{3}} = \frac{2e^{i\pi/3}}{2e^{-i\pi/3}} = \underline{\underline{e^{i2\pi/3}}}$$

$$\therefore \left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{13} = \left(e^{i2\pi/3}\right)^{13} = e^{i26\pi/3}.$$

Now $\frac{26\pi}{3} = \frac{13}{3}.2\pi = \left(4 + \frac{1}{3}\right).2\pi$

$= 4.2\pi + \frac{2\pi}{3}$

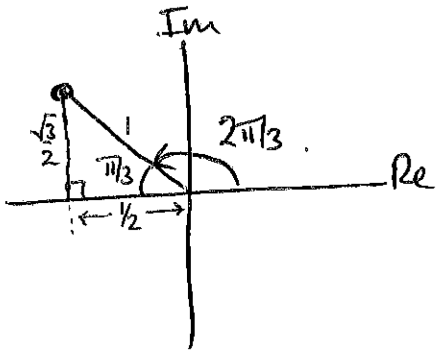
$$\therefore e^{i26\pi/3} = e^{i4.2\pi}\, e^{i2\pi/3} = e^{i2\pi/3}$$

since $e^{i4.2\pi} = \left(e^{i2\pi}\right)^4 = 1^4 = 1$

$$\therefore \left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^{13} = e^{i2\pi/3} = \underline{\underline{-\frac{1}{2} + i\frac{\sqrt{3}}{2}}}$$

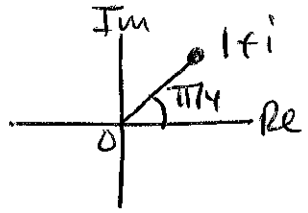
$\cos\frac{2\pi}{3} = -\frac{1}{2}$

$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$



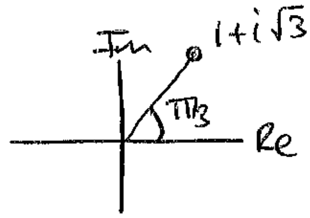
4) Let $z_1 = 1+i$

$|z_1| = \sqrt{1+1} = \sqrt{2}$, $\quad \operatorname{Arg} z_1 = \pi/4$



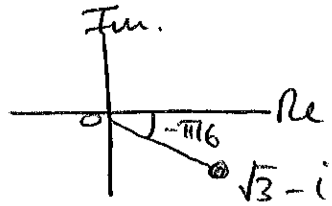
Let $z_2 = 1+i\sqrt{3}$

$|z_2| = \sqrt{1+3} = 2.$, $\quad \operatorname{Arg} z_2 = \pi/3.$



Let $z_3 = \sqrt{3} - i$

$|z_3| = \sqrt{3+1} = 2$, $\quad \operatorname{Arg} z_3 = -\pi/6.$



$\therefore\ z_1 = \sqrt{2}e^{i\pi/4}$; $z_2 = 2e^{i\pi/3}$; $z_3 = 2e^{-i\pi/6}.$

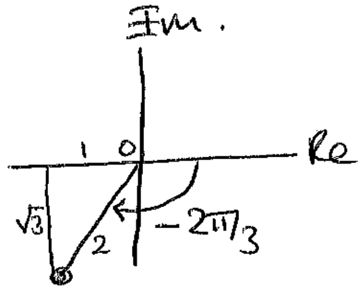
$$\therefore\ z = \frac{8\, z_1^6}{z_2^2\, z_3^3} = \frac{8.\sqrt{2}^6\ e^{i6\pi/4}}{2^2 e^{i2\pi/3}.\ 2^3.\ e^{-i3\pi/6}}.$$

$$= \frac{8.\ 2^3.\ e^{i3\pi/2}}{2^5\ e^{i2\pi/3}.\ e^{-i\pi/2}}$$

$$= 2.\ e^{i3\pi/2}\ e^{-i2\pi/3}\ e^{i\pi/2}$$

$$= 2\ e^{i4\pi/2}.\ e^{-i2\pi/3}$$

$$= 2e^{-i2\pi/3}.$$



$\therefore\ |z| = 2$, $\operatorname{Arg} z = -\dfrac{2\pi}{3}.$

5)(a) de Moivre's theorem states that

$$(\cos\theta + i\sin\theta)^q = \cos q\theta + i\sin q\theta$$

for any <u>rational</u> number q.

Note that $\cos 2A = 2\cos^2 A - 1$
$\sin 2A = 2\sin A\cos A$.

So we can write $\cos\theta = 2\cos^2\theta/2 - 1$
$\sin\theta = 2\sin\theta/2\cos\theta/2$.

$$\therefore \frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta} = \frac{2\cos^2\theta/2 + 2i\sin\theta/2\cos\theta/2}{2\cos^2\theta/2 - 2i\sin\theta/2\cos\theta/2}$$

$$= \frac{\cos\theta/2 + i\sin\theta/2}{\cos\theta/2 - i\sin\theta/2}.$$

$$= \frac{(\cos\theta/2 + i\sin\theta/2)^2}{\cos^2\theta/2 + \sin^2\theta/2}.$$

$$= (\cos\theta/2 + i\sin\theta/2)^2$$

since $\cos^2\theta/2 + \sin^2\theta/2 = 1$

$$\therefore \left(\frac{1+\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta}\right)^n = \left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)^{2n} = \underline{\cos n\theta + i\sin n\theta}$$

(using de Moivre's theorem).

(b) $\quad z_1 = e^{i\pi/3}, \quad z_2 = e^{-i\pi/4}.$

$$\xi_1 = \frac{z_1}{z_2^{\,2}} = \frac{e^{i\pi/3}}{e^{-i\pi/2}} = e^{i\pi/3} \cdot e^{i\pi/2}$$

$$= e^{i(\pi/3 + \pi/2)}$$

$$= e^{i5\pi/6}.$$

$$= \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$$

= [illegible]

$$\xi_2 = z_1^{\,2} z_2^{\,4} = e^{i2\pi/3} \cdot e^{-i\pi}$$

$$= e^{i(2\pi/3 - \pi)}$$

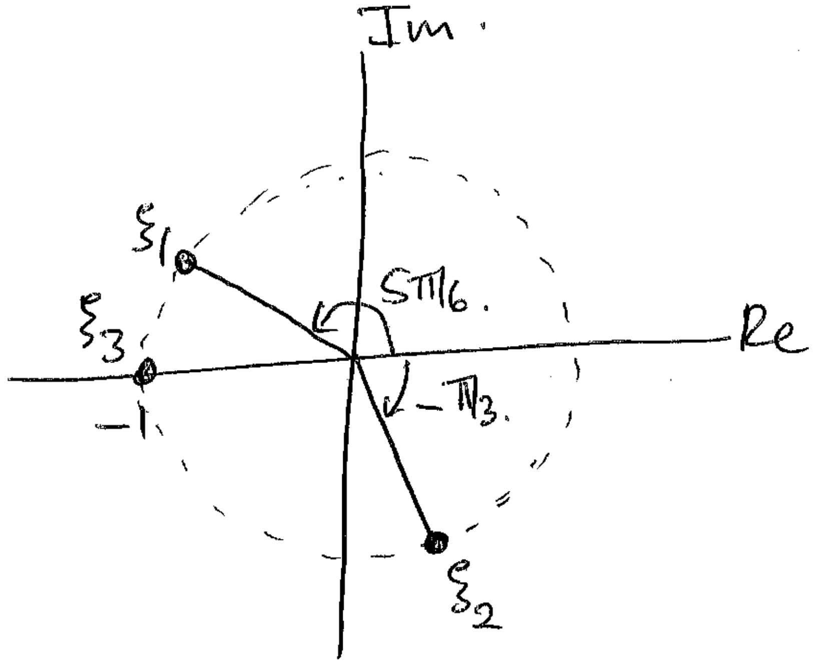
$$= e^{-i\pi/3}$$

$$= \cos(-\pi/3) + i\sin(-\pi/3)$$

$$= \cos(\pi/3) - i\sin(\pi/3)$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$\xi_3 = \frac{1}{z_2^{\,4}} = \frac{1}{e^{-i\pi}} = e^{i\pi} = \underline{-1}$$



(6) Let $z = re^{i\theta}$

then $\ln z = \ln(re^{i\theta})$

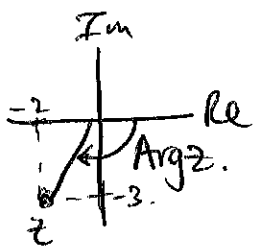
$= \ln r + \ln(e^{i\theta})$

$= \underline{\ln r + i\theta}$

Since $r = |z|$, $\theta = \arg z$

$\therefore \underline{\ln z = \ln|z| + i\arg z}$

If $z = -2-3i$, $|z| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$



$\operatorname{Arg} z = -\pi + \tan^{-1}(3/2)$ — principal argument

$= \underline{-2.16 \text{ radians}}.$

$\therefore \underline{\ln z = \ln\sqrt{13} + i(2n\pi - 2.16)}$, $n = 0, \pm 1, \pm 2, \ldots$