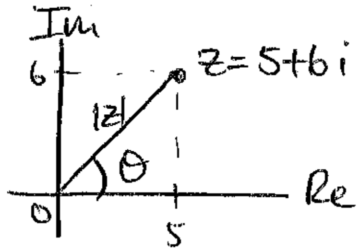
MAS140/151 Example Sheet 5 Solutions.

1). a) $(3+4i)+(2+2i) = (3+2)+(4i+2i)$
$$= 5+6i$$

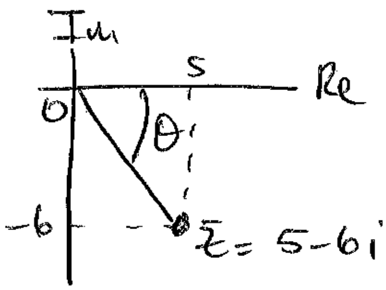
$\bar{z} = 5-6i$

$|z| = \sqrt{5^2+6^2} = \sqrt{25+36} = \sqrt{61} = |\bar{z}|$



$\theta = \operatorname{Arg} z \;:\; \tan\theta = \frac{6}{5}$

$\therefore \operatorname{Arg} z = \tan^{-1}\frac{5}{6}$ and in $(0, \pi/2)$.
$$= \underline{0.8761}$$

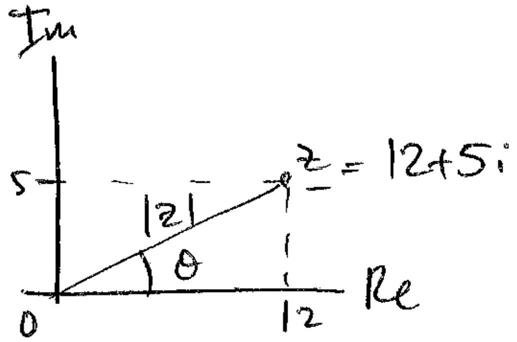


$\Rightarrow \operatorname{Arg}\bar{z} = \tan^{-1}\left(-\frac{5}{6}\right)$ and in $\left(-\frac{\pi}{2}, 0\right)$
$$= \underline{-0.8761}$$

b) $(2+3i)(3-2i) = 2.3 - 2.2i + 3.3i - 6i^2$
$$= 6 - 4i + 9i + 6 \quad (i^2=-1)$$
$$= 12+5i$$

$\therefore \bar{z} = 12-5i$

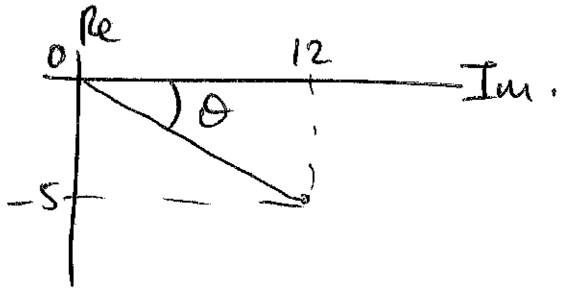
$|z| = \sqrt{12^2+5^2} = \sqrt{144+25} = \sqrt{169} = 13 = |\bar{z}|$



$$\operatorname{Arg} z = \theta : \tan\theta = \frac{5}{12}$$

$$\therefore \operatorname{Arg} z = \tan^{-1}\frac{5}{12} \quad \text{and in } (0, \pi/2)$$

$$= \underline{0.3948}$$



$$\operatorname{Arg} \bar{z} = \theta : \tan\theta = -\frac{5}{12}$$

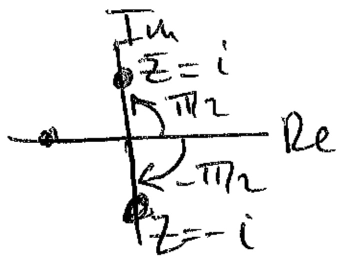
$$\operatorname{Arg} \bar{z} = \tan^{-1}\left(\frac{-5}{12}\right) \quad \text{and in } \left(-\frac{\pi}{2}, 0\right)$$

$$= \underline{-0.3948}$$

(c) $\frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

$\therefore \underline{z = -i} \Rightarrow \underline{\bar{z} = i}$

$|z| = \sqrt{1^2} = 1 = |\bar{z}|$



$$\therefore \operatorname{Arg} z = -\pi/2$$

$$\underline{\operatorname{Arg} \bar{z} = \pi/2}$$

(d) $z = \dfrac{(1+i)^2}{1-i} = \dfrac{(1+i)^3}{(1-i)(1+i)} = \dfrac{(1+i)^3}{2}$

now $(1+i)^3 = 1 + 3i + 3i^2 + i^3$

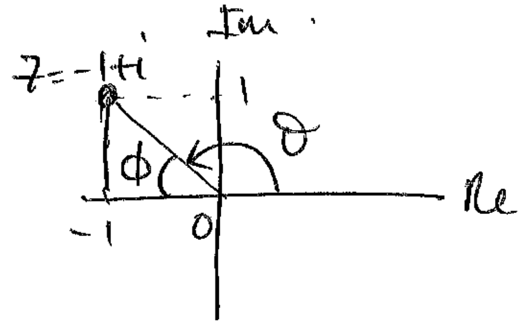
$= 1 + 3i - 3 - i$

$= -2 + 2i$

$\therefore\ z = \dfrac{(1+i)^3}{2} = \dfrac{-2+2i}{2} = \underline{-1+i}$

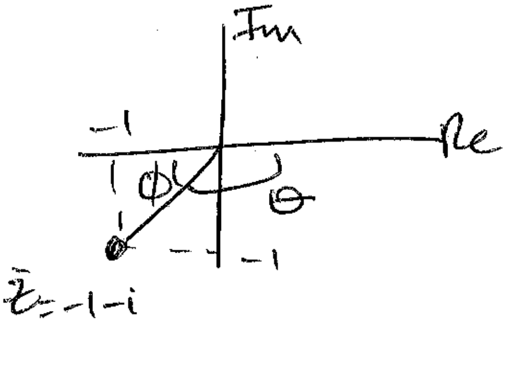
$\therefore\ \bar{z} = \underline{-1-i}$

$|z| = \sqrt{1^2+1^2} = \sqrt{2} = |\bar{z}|$



$\tan\phi = 1 \Rightarrow \phi = \pi/4$.

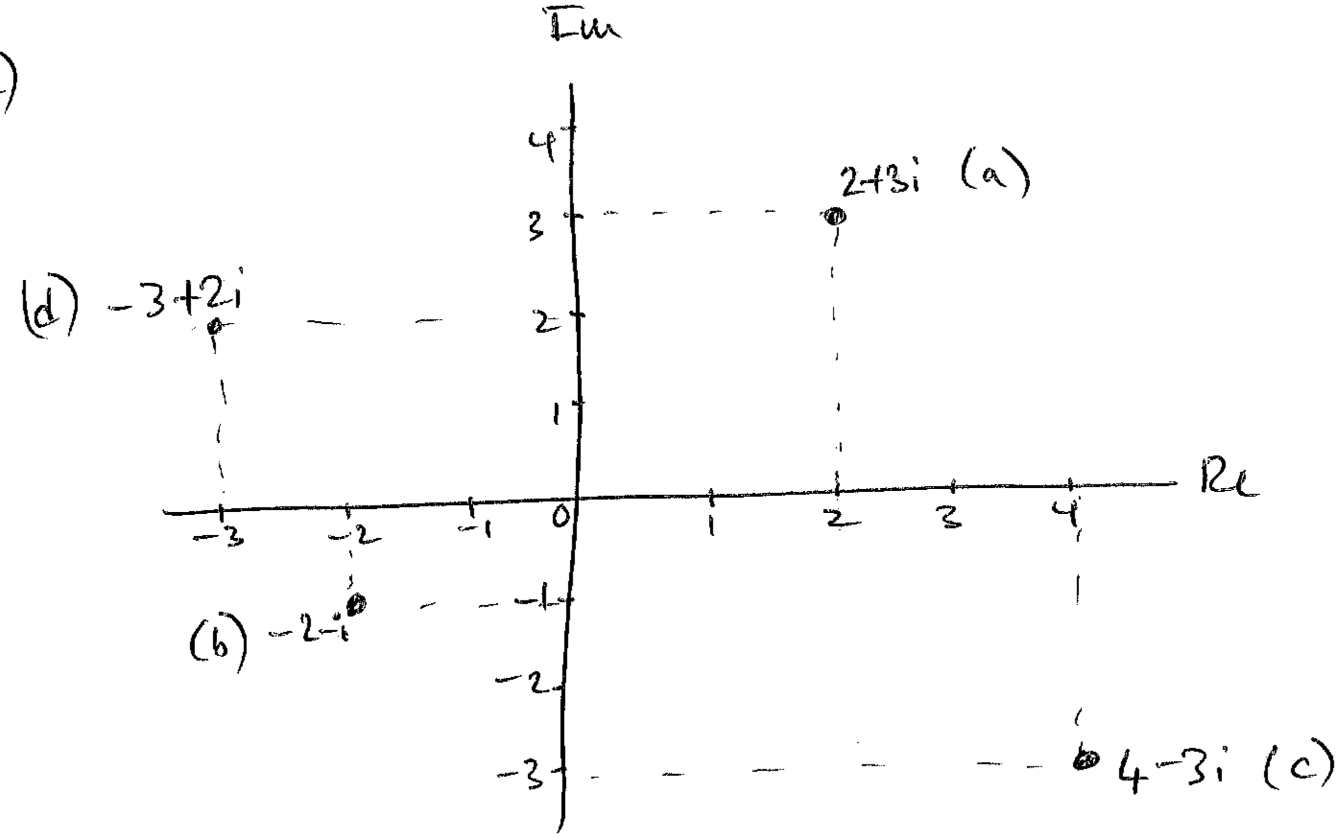
$\therefore\ \theta = \operatorname{Arg} z = \pi - \pi/4 = \underline{\underline{\dfrac{3\pi}{4}}}$



$\tan\phi = 1 \Rightarrow \phi = \pi/4$.

$\therefore\ \theta = \operatorname{Arg}\bar{z} = -\pi + \dfrac{\pi}{4} = \underline{\underline{-\dfrac{3\pi}{4}}}$

2)



3)

$$a+ib = \frac{x}{2+i} - \frac{2y}{1-i} + 3i$$

$$= \frac{x(2-i)}{(2+i)(2-i)} - \frac{2y(1+i)}{(1-i)(1+i)} + 3i$$

$$= \frac{2x - ix}{4+1} - \frac{2y + 2iy}{1+1} + 3i$$

$$= \frac{2x}{5} - \frac{ix}{5} - y - iy + 3i$$

$$= \left(\frac{2x}{5} - y\right) + \left(3 - \frac{x}{5} - y\right)i$$

$$\therefore \quad a = \frac{2x}{5} - y \quad ; \quad b = 3 - \frac{x}{5} - y$$

$\therefore \quad a + ib = 0 \Rightarrow a = 0$ and $b = 0$.

$$a = 0 \Rightarrow \frac{2x}{5} - y = 0 \Rightarrow y = \frac{2x}{5}.$$

$$b = 0 \Rightarrow 3 - \frac{x}{5} - y = 0$$

$$\Rightarrow 3 - \frac{y}{2} - y = 0$$

$$\Rightarrow 3 - \frac{3y}{2} = 0$$

$$\Rightarrow \frac{3y}{2} = 3 \Rightarrow \underline{\underline{y = 2}}.$$

$$\therefore \quad y = \frac{2x}{5} \Rightarrow x = \frac{5}{2} \cdot y \Rightarrow x = \frac{5}{2} \cdot 2 = 5$$

$$\therefore \quad \underline{\underline{x = 5, \; y = 2}}$$

4) Let $z = x + iy$

$\therefore\ z + 2i - 3 = (x-3) + (2+y)i$

and $z + 3i = x + (3+y)i$

So $|z+2i-3|^2 = (x-3)^2 + (2+y)^2$

and $|z+3i|^2 = x^2 + (3+y)^2$

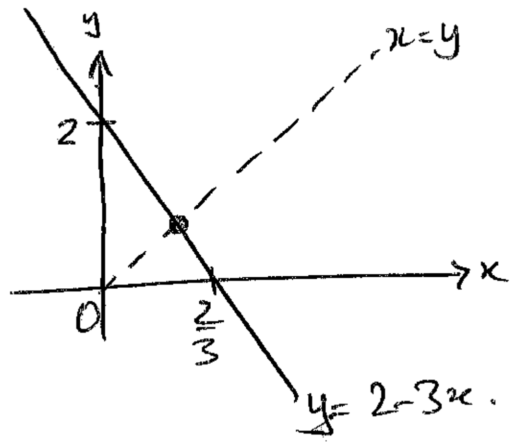
Equate $\Rightarrow (x-3)^2 + (2+y)^2 = x^2 + (3+y)^2$

$\therefore\ \cancel{x^2} - 6x + 9 + 4 + 4y + \cancel{y^2} = \cancel{x^2} + 9 + 6y + \cancel{y^2}$

$\therefore\ -6x + 13 + 4y = 9 + 6y$

$-6x + 4 = 2y$

$\therefore\ \underline{\underline{y = 2 - 3x}}$



If z is such that $\arg z = \pi/4$

then $\underline{\underline{x = y}}$

So the point we seek is at the intersection of the lines $y = 2-3x$ and $y = x$.

$\therefore\ x = 2 - 3x \Rightarrow 4x = 2 \Rightarrow \underline{\underline{x = 1/2}}$

$y = 2 - 3x \Rightarrow y = 2 - 3/2 = \underline{\underline{1/2}}$

$\therefore\ z = x + iy = \underline{\dfrac{1+i}{2}}$

5) $$z^3 - (2+i)z^2 + z - 2 - i = 0.$$

Note that $z = 2+i$ is a root.

$\therefore$ write

$$z^3 - (2+i)z^2 + z - 2 - i = (z-2-i)(z^2 + az + b)$$

Multiply out the R.H.S. :

$$(z-2-i)(z^2+az+b) = z^3 - (2+i)z^2 + az^2 - (2+i)az + bz - (2+i)b.$$

$$= z^3 + [a-(2+i)]z^2 + [b-(2+i)a]z - (2+i)b.$$

<u>Compare powers of z :</u>

z^0 : $-(2+i) = -(2+i)b \Rightarrow \underline{\underline{b=1}}.$

z^2 $-(2+i) = a-(2+i) \Rightarrow \underline{\underline{a=0}}.$

$\therefore z^3 - (2+i)z^2 + z - 2 - i = (z-2-i)(z^2+1)$

$\therefore$ Need to factorise z^2+1

$z^2+1=0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$

$\therefore$ The roots are $2+i, \pm i$.