Examples 4: Partial Differentiation

1. For each of the following functions, calculate $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Show that $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

   (a) $x^3 + 3x^2y + xy^2 + 4y^3$

   (b) $x \sin(xy)$

   (c) $xy^2 \ln(x^2 + y^2)$

   (d) $\dfrac{\sin r}{r}$, where $r^2 = x^2 + y^2$. [First show that $\partial r / \partial x = x/r$.]

2. If $f(x, y) = 3x^2y + y^3 + 4xy^2$ show that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 3f$.

3. If $f(x, y) = \cos^3(x^2 - y^2)$ find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. Show that $y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y} = 0$.

4. If $V = \dfrac{x}{t^{3/2}} \exp\left(\dfrac{x^2}{t}\right)$, show that the ratio of $\dfrac{\partial V}{\partial t}$ to $\dfrac{\partial^2 V}{\partial x^2}$ is a constant.

5. If $z = \ln \sqrt{x^2 + y^2}$, show that

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0.$$

   [This equation is called Laplace's equation in two dimensions. It arises in many physical and engineering applications.]

6. Show that if $r^2 = x^2 + y^2$ and $f(x, y) = F(r)$ then $\dfrac{\partial f}{\partial x} = \dfrac{F'(r)x}{r}$.

   Show further that $\dfrac{\partial^2 f}{\partial x^2} = F''(r)\dfrac{x^2}{r^2} + F'(r)\dfrac{y^2}{r^3}$ and hence that $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = F''(r) + F'(r)/r$.

Answers

1. (a) $\dfrac{\partial f}{\partial x} = 3x^2 + 6xy + y^2$, $\dfrac{\partial f}{\partial y} = 3x^2 + 2xy + 12y^2$

   (b) $\dfrac{\partial f}{\partial x} = \sin(xy) + xy\cos(xy)$, $\dfrac{\partial f}{\partial y} = x^2\cos(xy)$

   (c) $\dfrac{\partial f}{\partial x} = y^2\ln(x^2+y^2) + \dfrac{2x^2y^2}{x^2+y^2}$, $\dfrac{\partial f}{\partial y} = 2xy\ln(x^2+y^2) + \dfrac{2xy^3}{x^2+y^2}$

   (d) $\dfrac{\partial f}{\partial x} = \dfrac{x}{r^3}(r\cos r - \sin r)$, $\dfrac{\partial f}{\partial y} = \dfrac{y}{r^3}(r\cos r - \sin r)$

4. $\dfrac{\partial V}{\partial t} = -\exp\left(\dfrac{x^2}{t}\right)\left[\dfrac{3}{2}xt^{-5/2} + x^3t^{-7/2}\right] = -\dfrac{1}{4}\dfrac{\partial^2 V}{\partial x^2}$