Solutions to Partial Differentiation

1. (i) $\dfrac{\partial f}{\partial x} = 3x^2 + 6xy + y^2, \quad \dfrac{\partial f}{\partial y} = 3x^2 + 2xy + 12y^2$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(3x^2 + 2xy + 12y^2) = 6x + 2y$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(3x^2 + 6xy + y^2) = 6x + 2y$$

(ii) $\dfrac{\partial f}{\partial x} = x\dfrac{\partial}{\partial x}\sin(xy) + \sin(xy) = xy\cos(xy) + \sin(xy)$

$$\frac{\partial f}{\partial y} = x\frac{\partial}{\partial y}\sin(xy) = x^2\cos(xy)$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}[x^2\cos(xy)] = x^2\frac{\partial}{\partial x}\cos(xy) + 2x\cos(xy) = -x^2y\sin(xy) + 2x\cos(xy)$$
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}[xy\cos(xy) + \sin(xy)] = x[\cos(xy) - xy\sin(xy)] + x\cos(xy)$$
$$= 2x\cos(xy) - x^2y\sin(xy)$$

(iii)
$$\begin{aligned}
\frac{\partial f}{\partial x} &= y^2\left\{\ln(x^2+y^2) + x\frac{\partial}{\partial x}\ln(x^2+y^2)\right\} = y^2\ln(x^2+y^2) + xy^2\frac{2x}{x^2+y^2} \\
&= y^2\ln(x^2+y^2) + \frac{2x^2y^2}{x^2+y^2} \\
\frac{\partial f}{\partial y} &= x\left\{2y\ln(x^2+y^2) + y^2\frac{\partial}{\partial y}\ln(x^2+y^2)\right\} = 2xy\ln(x^2+y^2) + xy^2\frac{2y}{x^2+y^2} \\
&= 2xy\ln(x^2+y^2) + \frac{2xy^3}{x^2+y^2} \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left\{2xy\ln(x^2+y^2) + \frac{2xy^3}{x^2+y^2}\right\} \\
&= 2y\left\{\ln(x^2+y^2) + x\frac{\partial}{\partial x}\ln(x^2+y^2)\right\} + 2y^3\left\{\frac{(x^2+y^2) - x \times 2x}{(x^2+y^2)^2}\right\} \\
&= 2y\ln(x^2+y^2) + 2xy\frac{2x}{(x^2+y^2)} + \frac{2y^3}{(x^2+y^2)} - \frac{4x^2y^3}{(x^2+y^2)^2} \\
\frac{\partial^2 f}{\partial y \partial y} &= \frac{\partial}{\partial y}\left\{y^2\ln(x^2+y^2) + \frac{2x^2y^2}{x^2+y^2}\right\} \\
&= 2y\ln(x^2+y^2) + y^2\frac{\partial}{\partial y}\ln(x^2+y^2) + 2x^2\frac{(x^2+y^2)2y - y^2 \times 2y}{(x^2+y^2)^2} \\
&= 2y\ln(x^2+y^2) + y^2\frac{2y}{(x^2+y^2)} + \frac{4x^2y}{(x^2+y^2)} - \frac{4x^2y^3}{(x^2+y^2)^2} \\
&= \frac{\partial^2 f}{\partial x \partial y}
\end{aligned}$$

(iv) $\dfrac{\partial f}{\partial x} = \dfrac{d}{dr}\left(\dfrac{\sin r}{r}\right)\dfrac{\partial r}{\partial x} = \dfrac{r\cos r - \sin r}{r^2}\dfrac{\partial r}{\partial x}$. Now $r^2 = x^2 + y^2$ and so $2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$ and similarly $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$.

Hence $\dfrac{\partial f}{\partial x} = \dfrac{x}{r^3}(r\cos r - \sin r)$ and $\dfrac{\partial f}{\partial y} = \dfrac{y}{r^3}(r\cos r - \sin r)$. Thus

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\left\{\frac{y}{r^3}(r\cos r - \sin r)\right\} = y\frac{d}{dr}\left\{\frac{(r\cos r - \sin r)}{r^3}\right\}\frac{\partial r}{\partial x} \\
&= \frac{xy}{r}\frac{[r^3(\cos r - r\sin r - \cos r) - (r\cos r - \sin r)3r^2]}{r^6} \\
&= \frac{xy}{r^4}(-r\sin r) - \frac{3xy}{r^5}(r\cos r - \sin r) = -\frac{xy}{r^3}\sin r - \frac{3xy}{r^5}(r\cos r - \sin r)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\left\{\frac{x}{r^3}(r\cos r - \sin r)\right\} = x\frac{d}{dr}\left\{\frac{(r\cos r - \sin r)}{r^3}\right\}\frac{\partial r}{\partial y} \\
&= \frac{xy}{r}\frac{d}{dr}\left\{\frac{(r\cos r - \sin r)}{r^3}\right\} \\
&= \frac{\partial^2 f}{\partial x \partial y} \quad \text{as before.}
\end{aligned}$$

2. $f(x,y) = 3x^2y + y^3 + 4xy^2 \Rightarrow \dfrac{\partial f}{\partial x} = 6xy + 4y^2$ and $\dfrac{\partial f}{\partial y} = 3x^2 + 3y^2 + 8xy$. Hence
$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = x(6xy + 4y^2) + y(3x^2 + 3y^2 + 8xy) = 9x^2y + 12xy^2 + 3y^3 = 3f.$

3. $f(x,y) = \cos^3(x^2 - y^2)$
$\Rightarrow \dfrac{\partial f}{\partial x} = 3\cos^2(x^2 - y^2)[-\sin(x^2 - y^2)](2x)$ and
$\dfrac{\partial f}{\partial y} = 3\cos^2(x^2 - y^2)[-\sin(x^2 - y^2)](-2y)$. Hence $y\dfrac{\partial f}{\partial x} + x\dfrac{\partial f}{\partial y} = 0.$

4. $V = xt^{-3/2}\exp(x^2/t) \Rightarrow \dfrac{\partial V}{\partial t} = x\left(-\dfrac{3}{2}\right)t^{-5/2}\exp(x^2/t) + xt^{-3/2}\exp(x^2/t)\left(-\dfrac{x^2}{t^2}\right)$ or
$\dfrac{\partial V}{\partial t} = -xt^{-7/2}\exp(x^2/t)(3t + 2x^2)/2$
Also $\dfrac{\partial V}{\partial x} = t^{-3/2}\exp(x^2/t) + t^{-3/2}x\exp(x^2/t)\left(\dfrac{2x}{t}\right) = t^{-5/2}(t + 2x^2)\exp(x^2/t)$
$\Rightarrow \dfrac{\partial^2 f}{\partial x^2} = t^{-5/2}\exp(x^2/t)\left[(4x) + (t + 2x^2)\left(\dfrac{2x}{t}\right)\right] = 2xt^{-7/2}\exp(x^2/t)(3t + 2x^2).$
Hence $\dfrac{\partial V}{\partial t}\Big/\dfrac{\partial^2 V}{\partial x^2} = -\dfrac{1}{4}.$

5. $z = \ln r$ where $r = \sqrt{x^2 + y^2} \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, see 1. (iv). Thus
$\dfrac{\partial z}{\partial x} = \dfrac{1}{r}\dfrac{\partial r}{\partial x} = \dfrac{x}{r^2} \Rightarrow \dfrac{\partial^2 z}{\partial x^2} = \dfrac{1}{r^2} - \dfrac{2x}{r^3}\dfrac{\partial r}{\partial x} = \dfrac{1}{r^2} - \dfrac{2x^2}{r^4}.$
Likewise, $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{1}{r^2} - \dfrac{2y^2}{r^4}$ and so
$\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = \dfrac{2}{r^2} - \dfrac{(2x^2 + 2y^2)}{r^4} = 0.$

6. Because F only depends upon r, we can write $\dfrac{\partial f}{\partial x} = \dfrac{dF}{dr}\dfrac{\partial r}{\partial x} = F'(r)\dfrac{x}{r}$ (see $1(iv)$).

Hence $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(F'(r)\dfrac{x}{r}\right) = \dfrac{\partial F'(r)}{\partial x}\left(\dfrac{x}{r}\right) + \dfrac{F'(r)}{r}\dfrac{\partial x}{\partial x} + F'(r)x\dfrac{\partial}{\partial x}\left(\dfrac{1}{r}\right)$

$= \left(F''(r)\dfrac{x}{r}\right)\left(\dfrac{x}{r}\right) + \dfrac{F'(r)}{r} + F'(r)x\left(\dfrac{-1}{r^2}\dfrac{x}{r}\right)$

$= F''(r)\dfrac{x^2}{r^2} + F'(r)\dfrac{1}{r} - F'(r)\dfrac{x^2}{r^3} = F''(r)\dfrac{x^2}{r^2} + F'(r)\dfrac{y^2}{r^3}$

because $r^2 - x^2 = y^2$.

Likewise $\dfrac{\partial^2 f}{\partial y^2} = F''(r)\dfrac{y^2}{r^2} + F'(r)\dfrac{x^2}{r^3}$

and so $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = F''(r)\dfrac{(x^2+y^2)}{r^2} + F'(r)\dfrac{(y^2+x^2)}{r^3} = F''(r) + \dfrac{F'(r)}{r}$