Examples 3: Maclaurin and Taylor Series: L'Hôpital's Rule

1. Find the Maclaurin series for

   (a) $f(x) = \sin(3x + 2)$ (first 4 non-zero terms),

   (b) $f(x) = \sin^2 x$ (first 3 non-zero terms).

2. Show graphically that the equation $3x = e^{-2x}$ has exactly one real solution x_* and that $0 < x_* < \frac{1}{3}$. Use the first two and three terms of the Maclaurin series for e^{-2x} to derive approximations x_1 and x_2 for x_*. Given that the true value of x_* is 0.2163, find the percentage errors of your approximations.

3. Use Taylor's theorem to find the first 4 terms of the expansions of

   (a) $f(x) = \sqrt{x}$ about $x = 4$,

   (b) $f(x) = \sin x$ about $x = \pi/4$.

4. Show graphically that the equation $2 \sin x = x$ has one solution x_* satisfying $x_* > 0$. By using the first two non-zero terms of the Taylor expansion of $\sin x$ around $x = \frac{\pi}{2}$, derive an approximate value for x_*. Given that the true value of x_* is 1.8955 (4 d.p.), find the percentage error of your approximation.

5. Evaluate the following limits:

   (a) $\lim_{x \to 2} \dfrac{2x^2 + 3x - 14}{x - 2}$

   (b) $\lim_{x \to 0} \dfrac{\tan x - x}{x^3}$

   (c) $\lim_{x \to \infty} x^3 e^{-x}$

   (d) $\lim_{x \to 0+} x^p \ln x$ (p is a positive constant)

   (e) $\lim_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x} \right)$

   (f) $\lim_{x \to -1} \dfrac{\sin \pi x}{1 + x}$

Answers

1. (a) $\sin 2 + 3x \cos 2 - \dfrac{9x^2}{2} \sin 2 - \dfrac{9x^3}{2} \cos 2 + \dots$

   (b) $x^2 - \dfrac{1}{3}x^4 + \dfrac{2}{45}x^6 + \dots$

2. $x_1 = 0.2, \quad x_2 = 0.2192$. Percentage errors are -7.5% and 1.34%.

3. (a) $2 + \dfrac{(x-4)}{4} - \dfrac{(x-4)^2}{64} + \dfrac{(x-4)^3}{512} + \dots$

   (b) $\dfrac{1}{\sqrt{2}} \left\{ 1 + \left(x - \dfrac{\pi}{4} \right) - \dfrac{1}{2} \left(x - \dfrac{\pi}{4} \right)^2 - \dfrac{1}{6} \left(x - \dfrac{\pi}{4} \right)^3 + \dots \right\}$

4. $x_* \approx 1.8949$. 0.03% error.

5. (a) 11, (b) $\dfrac{1}{3}$, (c) 0, (d) 0, (e) 0, (f) $-\pi$.