MAS152 EXAMPLE SHEET 3 SOLUTIONS

1) (a)

$$f(x) = \sin(3x+2) \Rightarrow f(0) = \sin 2$$
$$f'(x) = 3\cos(3x+2) \Rightarrow f'(0) = 3\cos 2$$
$$f''(x) = -9\sin(3x+2) \Rightarrow f''(0) = -9\sin 2$$
$$f'''(x) = -27\cos(3x+2) \Rightarrow f'''(0) = -27\cos 2$$

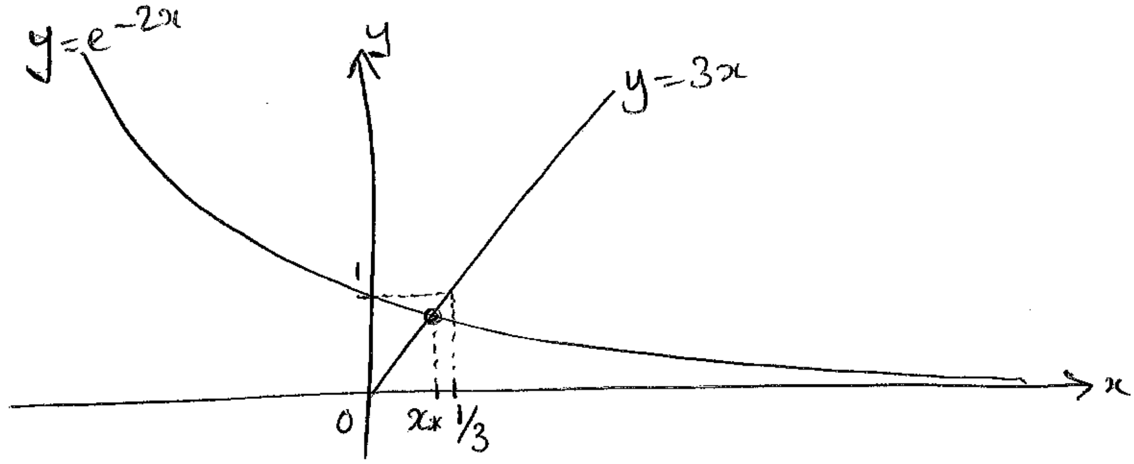
$$\therefore \sin(3x+2) = \sin 2 + 3x\cos 2 - \frac{9x^2}{2}\sin 2 - \frac{9x^3}{2}\cos 2 + \cdots$$

(b)

$$f(x) = \sin^2 x \Rightarrow f(0) = 0$$
$$f'(x) = 2\sin x\cos x \Rightarrow f'(0) = 0$$
$$= \sin 2x \quad \text{[makes subsequent calculation easier]}$$
$$f''(x) = 2\cos 2x \Rightarrow f''(0) = 2$$
$$f'''(x) = -4\sin 2x \Rightarrow f'''(0) = 0$$
$$f^{(4)}(x) = -8\cos 2x \Rightarrow f^{(4)}(0) = -8$$
$$f^{(5)}(x) = 16\sin 2x \Rightarrow f^{(5)}(0) = 0$$
$$f^{(6)}(x) = 32\cos 2x \Rightarrow f^{(6)}(0) = 32$$

$$\therefore \sin^2 x = \frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{32x^6}{6!} + \cdots = x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \cdots$$

2)



Since $e^{-2x_*} < 1 \Rightarrow x_* < \frac{1}{3}$ (and $x_* > 0$)

Let $f(x) = e^{-2x} \Rightarrow f(0) = 1$

$\therefore f'(x) = -2e^{-2x} \Rightarrow f'(0) = -2$

$f''(x) = 4e^{-2x} \Rightarrow f''(0) = 4$

$$\therefore e^{-2x} = 1 - 2x + \frac{4x^2}{2!} - \cdots$$

$$= \underline{1 - 2x + 2x^2 - \cdots}$$ (Maclaurin series)

<u>1st 2 terms</u>: $e^{-2x} \simeq 1-2x \Rightarrow 3x_* \simeq 1 - 2x_*$

$\Rightarrow \underline{\underline{x_* \simeq 0.2}} = x_1$

<u>1st 3 terms</u>: $e^{-2x} \simeq 1-2x+2x^2 \Rightarrow 3x_* \simeq 1-2x_*+2x_*^2$

$\Rightarrow 2x_*^2 - 5x_* + 1 \simeq 0$

$\Rightarrow x_* \simeq \frac{5-\sqrt{17}}{4} = 0.2192$

$\therefore \underline{x_2 = 0.2192}$

Actual value $= x_* = 0.2163$ (given)

<u>% age errors</u>: $100 \times \frac{(x_1 - x_*)}{x_*} = \underline{\underline{-7.5\%}}$; $100 \times \frac{(x_2 - x_*)}{x_*} = \underline{\underline{1.34\%}}$

3) (a)

$$f(x) = \sqrt{x} \Rightarrow f(4) = \sqrt{4} = \underline{2}$$

$$f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(4) = \frac{1}{2}\cdot\frac{1}{2} = \underline{\frac{1}{4}}$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(4) = -\frac{1}{4}\cdot\frac{1}{2^3} = \underline{-\frac{1}{32}}$$

$$f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(4) = \frac{3}{8}\cdot\frac{1}{2^5} = \underline{\frac{3}{256}}$$

$$\therefore \sqrt{x} = 2 + \frac{1}{4}(x-4) - \frac{1}{32}\frac{(x-4)^2}{2!} + \frac{3}{256}\frac{(x-4)^3}{3!} + \cdots$$

$$= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3 + \cdots$$

(b)

$$f(x) = \sin x \Rightarrow f(\pi/4) = 1/\sqrt{2}$$

$$f'(x) = \cos x \Rightarrow f'(\pi/4) = 1/\sqrt{2}$$

$$f''(x) = -\sin x \Rightarrow f''(\pi/4) = -1/\sqrt{2}$$

$$f'''(x) = -\cos x \Rightarrow f'''(\pi/4) = -1/\sqrt{2}$$

$$\therefore \sin x = \frac{1}{\sqrt{2}}\left[1 + \left(x-\frac{\pi}{4}\right) - \frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(x-\frac{\pi}{4}\right) - \frac{1}{2}\left(x-\frac{\pi}{4}\right)^2 - \frac{1}{6}\left(x-\frac{\pi}{4}\right)^3 + \cdots\right]$$

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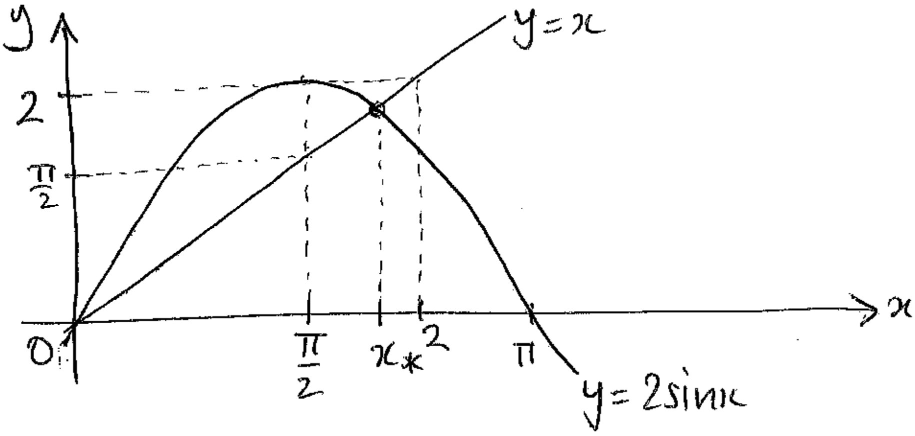
4)



let $f(x) = \sin x \quad \Rightarrow \quad f(\pi/2) = 1$

$f'(x) = \cos x \quad \Rightarrow \quad f'(\pi/2) = 0$

$f''(x) = -\sin x \quad \Rightarrow \quad f''(\pi/2) = -1$

$\therefore \sin x = 1 - \frac{1}{2}(x - \pi/2)^2 + \cdots$

Use 1st 2 terms as an approximation for sinx :

$2\sin x_* = x_* \Rightarrow 2 - \left(x_* - \frac{\pi}{2}\right)^2 \simeq x_*.$

$\therefore \left(x_* - \frac{\pi}{2}\right)^2 + x_* - 2 \simeq 0$

$\therefore \left(x_* - \frac{\pi}{2}\right)^2 + \left(x_* - \frac{\pi}{2}\right) + \left(\frac{\pi}{2} - 2\right) \simeq 0$

$\therefore x_* - \frac{\pi}{2} \simeq -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\left(\frac{\pi}{2} - 2\right)} = -\frac{1}{2} + \frac{1}{2}\sqrt{9 - 2\pi}$

$\therefore \underline{x_* \simeq 1.8949}$ (4 d.p.)

Actual value $x_* = 1.8955$

% age error $= \left(\frac{1.8949 - 1.8955}{1.8955}\right) \times 100 = \underline{\underline{-0.03\%}}.$

5) (a) $\lim_{x\to 2}\left[\frac{2x^2+3x-14}{x-2}\right] = \lim_{x\to 2}\left[\frac{4x+3}{1}\right] = \underline{11}$

(b) $\lim_{x\to 0}\left[\frac{\tan x - x}{x^3}\right] = \lim_{x\to 0}\left[\frac{\sec^2 x - 1}{3x^2}\right] = \lim_{x\to 0}\left[\frac{2\tan x\sec^2 x}{6x}\right]$

$= \lim_{x\to 0}\left[\frac{4\tan^2 x\sec^2 x + 2\sec^4 x}{6}\right] = \underline{\frac{1}{3}}$

(c) $\lim_{x\to\infty}\left[x^3 e^{-x}\right] = \lim_{x\to\infty}\left[\frac{x^3}{e^x}\right] = \lim_{x\to\infty}\left[\frac{3x^2}{e^x}\right]$

$= \lim_{x\to\infty}\left[\frac{6x}{e^x}\right] = \lim_{x\to\infty}\left[\frac{6}{e^x}\right] = \underline{0}.$

(d) $\lim_{x\to 0+}(x^p \ln x) = \lim_{x\to 0+}\left[\frac{\ln x}{x^{-p}}\right] = \lim_{x\to 0+}\left[\frac{1/x}{-px^{-p-1}}\right]$

$= \lim_{x\to 0+}\left[\frac{x^p}{-p}\right] = \underline{0} \quad (p>0)$

(e) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right) = \lim_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right) = \lim_{x\to 0}\left(\frac{\cos x - 1}{x\cos x + \sin x}\right)$

$= \lim_{x\to 0}\left(\frac{-\sin x}{2\cos x - x\sin x}\right) = \underline{0}$

(f) $\lim_{x\to -1}\left(\frac{\sin \pi x}{1+x}\right) = \lim_{x\to -1}\left(\frac{\pi\cos\pi x}{1}\right) = \underline{-\pi}$