Examples 2: Differentiation of functions of a single variable

1. Find the derivatives of the following functions by working from first principles:
   (i) x^n (where $n \neq 1$ is a constant), (ii) $\dfrac{1}{x}$, (iii) $\tan x$.

2. Use either the product or quotient rule to differentiate each of the following with respect to x: (i) $\tan x$, (ii) $\sec x$, (iii) $x^2 e^x$, (iv) $\dfrac{x^2-1}{x^2+1}$, (v) $\dfrac{\ln x}{x}$.

3. Use the function-of-a-function (chain) rule to differentiate each of the following:
   (i) $\sin^3 x$, (ii) $\cos\sqrt{x}$, (iii) $\tan(mx+a)$ (where a and m are constants),
   (iv) e^{x^3+2x}, (v) $\ln(\sin x)$.

4. Differentiate each of the following functions with respect to x:

   (i) $\cos^{-1} x$ (ii) $\sin^{-1}(2x^2)$ (iii) $\ln(\sin\sqrt{x})$

   (iv) $\tan^{-1} x$ (v) $(x^2+1)^2 \tan^{-1} x$ (vi) $\ln\cos\left(\dfrac{1}{x}\right)$

   (vii) $\ln(\ln x)$ (viii) $\ln\left(e^{1/x}\right)$ (ix) $\sqrt{\dfrac{1-x}{1+x}}$

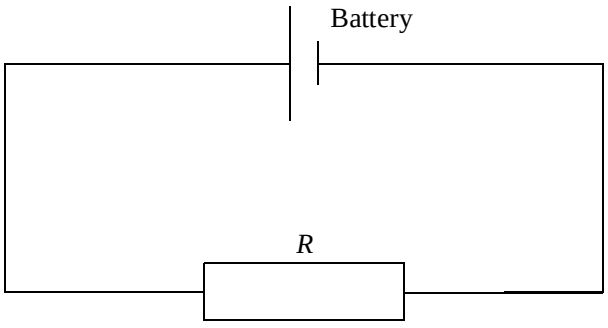
5. If y is related to x by $y = x^2 + 2x + \cosh^{-1}(x^2+1)$ and if x is related to t by $x = 2 + \sin t$, find the value of $\dfrac{dy}{dt}$ when $t = \pi$.

6. If $y = x\sin x + \ln x$, find $\dfrac{d^2x}{dy^2}$.

7. If $\dfrac{dy}{dx} = P(x)$, show that $\dfrac{d^2x}{dy^2} = -\dfrac{P'}{P^3} = -\dfrac{d^2y}{dx^2}\Big/\left(\dfrac{dy}{dx}\right)^3$.

8. What are the greatest and least values (*not* maximum and minimum) attained by the function $x^3 - 5x^2 + 3x$ when x is constrained to lie in the interval $[0,5]$?

9. The power output, P, in watts of a battery in the simple circuit pictured below is $P = V^2R/(r+R)^2$, where V the voltage of the battery and r its internal resistance are known values. Show that the maximum power occurs when $R = r$.



10. Determine the stationary points of $y = x^2 \exp(-2x)$. Show that the curve has two points of inflexion. Sketch the curve.

11. A rectangular field has to contain a certain area and is to be fenced off along a straight river. If no fencing is needed along the river, show that the minimum amount of fencing will be required when the length of the field is twice its width.

12. A river estuary is triangular in shape, being a mile in width at its mouth and 10 miles along each bank. A coast road (which will join the two points at the mouth on opposite banks) is to be built by roads along each bank and joined by a bridge. The road will cost £10,000 per mile. A bridge of length h miles costs £h^2 million.
Find the cost of the project for a bridge parallel to the mouth of the estuary in terms of the distance x in miles of the ends of the bridge from the apex of the triangle and show that the minimum cost is £190,000.

13. Two passages meet at right angles and have widths a and b. Show that the longest ladder that can be taken around the corner (while being kept horizontal) has length
$$(a^{2/3} + b^{2/3})^{3/2}.$$

14. Evaluate the derivatives of each of the following functions at $x = 0$:
(i) $(1 + x^2) \tan^{-1} x$, (ii) $\exp[x \ln\{\ln(3x + 2)\}]$, (iii) $\dfrac{x(1 - x^2)^2}{\sqrt{1 + x^2}}$.

ANSWERS

1. (i) nx^{n-1} (ii) $-1/x^2$ (iii) $\sec^2 x$
Hint: use the Binomial theorem for (i) and (ii); use the formula for $\tan(A + B)$ for (iii).

2. (i) $\sec^2 x$ (ii) $\sec x \tan x$ (iii) $x(2 + x)e^x$ (iv) $4x/(x^2 + 1)^2$ (v) $(1 - \ln x)/x^2$

3. (i) $3 \sin^2 x \cos x$ (ii) $-\sin \sqrt{x}/(2\sqrt{x})$ (iii) $m \sec^2(mx + a)$ (iv) $(3x^2 + 2)e^{x^3+2x}$ (v) $\cot x$

4. (i) $-1/\sqrt{1 - x^2}$ (ii) $4x/\sqrt{1 - 4x^4}$ (iii) $\cot(\sqrt{x})/(2\sqrt{x})$ (iv) $1/(1 + x^2)$

   (v) $(x^2 + 1)(1 + 4x \tan^{-1} x)$ (vi) $x^{-2} \tan(1/x)$ (vii) $1/(x \ln x)$ (viii) $-x^{-2}$

   (ix) $-(1 + x)^{-3/2}(1 - x)^{-1/2}$

5. -6.8165

6. $\dfrac{(-2x^2 \cos x + x^3 \sin x + 1)x}{(x \sin x + x^2 \cos x + 1)^3}$

8. Least value -9 . Greatest value 15.

10. Stationary points at $x = 0$ and $x = 1$. Points of inflexion at $x = 1 \pm 1/\sqrt{2}$.

14. (i) 1, (ii) $\ln(\ln 2)$, (iii) 1.