Solutions to Differentiation

1. (*i*) $f(x) = x^n$ and $f(x+h) = (x+h)^n$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^n - x^n}{h} = \left(\frac{x^n}{h}\right)\left\{\left(1+\frac{h}{x}\right)^n - 1\right\}\\ &= \left(\frac{x^n}{h}\right)\left\{\left[1+n\left(\frac{h}{x}\right)+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+\ldots\right]-1\right\}\\ &= nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\ldots\\ &\to nx^{n-1} \quad \text{as} \quad h\to 0.\end{aligned}$$

Hence $f'(x) = nx^{n-1}$.

(*ii*) This can be deduced from the last problem with $n = -1$. However, it is very easy to do it directly.

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \left(\frac{1}{h}\right)\left\{\frac{1}{x+h}-\frac{1}{x}\right\}\\ &= \left(\frac{1}{h}\right)\frac{[x-(x+h)]}{x(x+h)}\\ &= -\frac{1}{x(x+h)} \to -\frac{1}{x^2} \quad \text{as} \quad h\to 0.\end{aligned}$$

Hence $f'(x) = -x^{-2}$.

(*iii*) $f(x) = \tan x$ and $f(x+h) = \tan(x+h)$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[\tan(x+h)-\tan x]}{h}\\ &= \left(\frac{1}{h}\right)\left\{\frac{\tan x+\tan h}{1-\tan x\tan h}-\tan x\right\}\\ &= \frac{[(\tan x+\tan h)-\tan x(1-\tan x\tan h)]}{h(1-\tan x\tan h)}\\ &= \frac{\tan h}{h}\frac{1}{(1-\tan x\tan h)}(1+\tan^2 x)\\ &\to 1\times 1\times \sec^2 x = \sec^2 x \quad \text{as} \quad h\to 0\end{aligned}$$

2. (*i*)

$$\begin{aligned}\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\frac{d}{dx}\sin x - \sin x\frac{d}{dx}\cos x}{\cos^2 x}\\ &= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\end{aligned}$$

(*ii*)

$$\begin{aligned}\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x\frac{d}{dx}1 - 1\frac{d}{dx}\cos x}{\cos^2 x}\\ &= \frac{\sin x}{\cos^2 x} = \sec x\tan x\end{aligned}$$

(*iii*) $$\frac{d}{dx}x^2e^x = x^2\frac{d}{dx}e^x + e^x\frac{d}{dx}x^2 = x^2e^x + 2xe^x = xe^x(x+2)$$

(*iv*) $$\begin{aligned}\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) &= \frac{(x^2+1)\frac{d}{dx}(x^2-1)-(x^2-1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\ &= \frac{2x(x^2+1)-2x(x^2-1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}\end{aligned}$$

(*v*) $$\frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x\frac{d}{dx}\ln x - \ln x\frac{d}{dx}x}{x^2} = \frac{x\frac{1}{x}-\ln x}{x^2} = \frac{1-\ln x}{x^2}$$

3. (*i*) $$\begin{aligned}\frac{d}{dx}\sin^3 x &= \frac{d}{dx}u^3 \quad \text{where} \quad u=\sin x \\ &= \frac{d}{du}u^3\frac{d}{dx}u = 3u^2\cos x = 3\sin^2 x\cos x\end{aligned}$$

(*ii*) $$\begin{aligned}\frac{d}{dx}\cos(x^{1/2}) &= \frac{d}{dx}\cos u \quad \text{where} \quad u=x^{1/2} \\ &= \frac{d}{du}\cos u\frac{d}{dx}u = -\sin u\left(\frac{x^{-1/2}}{2}\right) = -\frac{\sin(x^{1/2})}{2x^{1/2}}\end{aligned}$$

(*iii*) $$\begin{aligned}\frac{d}{dx}\tan(mx+a) &= \frac{d}{dx}\tan u \quad \text{where} \quad u=mx+a \\ &= \frac{d}{du}\tan u\frac{d}{dx}u = m\sec^2 u = m\sec^2(mx+a)\end{aligned}$$

(*iv*) $$\begin{aligned}\frac{d}{dx}\exp(x^3+2x) &= \frac{d}{dx}\exp u \quad \text{where} \quad u=x^3+2x \\ &= \frac{d}{du}\exp u\frac{d}{dx}u = (\exp u)(3x^2+2) = (3x^2+2)\exp(x^3+2x)\end{aligned}$$

(*v*) $$\begin{aligned}\frac{d}{dx}\ln\sin x &= \frac{d}{dx}\ln u \quad \text{where} \quad u=\sin x \\ &= \frac{d}{du}\ln u\frac{d}{dx}u = \frac{1}{u}\cos x = \frac{\cos x}{\sin x} = \cot x\end{aligned}$$

4. (*i*) Set $y=\cos^{-1}x$ so that $x=\cos y$ $(0\le y\le\pi)$. Then
$$\frac{dx}{dy} = -\sin y = -\sqrt{1-\cos^2 y} = -\sqrt{1-x^2} \quad (\text{since } \sin y\ge 0 \text{ for } 0\le y\le\pi)$$
and so $\dfrac{dy}{dx} = 1\Big/\dfrac{dx}{dy} = \dfrac{-1}{\sqrt{1-x^2}}$

(*ii*) $$\begin{aligned}\frac{d}{dx}\sin^{-1}(2x^2) &= \frac{d}{dx}\sin^{-1}u \quad \text{where} \quad u=2x^2 \\ &= \frac{d}{du}\sin^{-1}u\frac{d}{dx}u = \frac{1}{\sqrt{1-u^2}}4x = \frac{4x}{\sqrt{1-4x^4}}\end{aligned}$$

(iii)
$$\begin{aligned}\frac{d}{dx}\ln[\sin(\sqrt{x})] &= \frac{d}{dx}\ln v \quad \text{where} \quad v=\sin u \text{ and } u=\sqrt{x}\\ &= \frac{d\ln v}{dv}\frac{dv}{du}\frac{du}{dx} = \frac{1}{v}\cos u\frac{x^{-1/2}}{2}\\ &= \frac{1}{\sin\sqrt{x}}\cos\sqrt{x}\frac{1}{2\sqrt{x}} = \frac{\cot\sqrt{x}}{2\sqrt{x}}\end{aligned}$$

(iv) Set $y=\tan^{-1}x$ so that $x=\tan y$. Then $\dfrac{dx}{dy}=\sec^2 y = 1+\tan^2 y = 1+x^2$ and so $\dfrac{dy}{dx} = 1\bigg/\dfrac{dx}{dy} = \dfrac{1}{1+x^2}$.

(v)
$$\begin{aligned}\frac{d}{dx}(x^2+1)^2\tan^{-1}x &= (x^2+1)^2\frac{d}{dx}\tan^{-1}x + \tan^{-1}x\frac{d}{dx}(x^2+1)^2\\ &= (x^2+1)^2\frac{1}{1+x^2} + \tan^{-1}x \times 2(x^2+1)2x\\ &= (x^2+1)+4x(x^2+1)\tan^{-1}x = (x^2+1)[1+4x\tan^{-1}x]\end{aligned}$$

(vi)
$$\begin{aligned}\frac{d}{dx}\ln\left[\cos\left(\frac{1}{x}\right)\right] &= \frac{d}{dx}\ln v \quad \text{where} \quad v=\cos u \text{ and } u=x^{-1}\\ &= \frac{d\ln v}{dv}\frac{dv}{du}\frac{du}{dx} = \frac{1}{v}(-\sin u)(-x^{-2})\\ &= \frac{1}{\cos(x^{-1})}\sin(x^{-1})x^{-2} = \frac{1}{x^2}\tan\left(\frac{1}{x}\right)\end{aligned}$$

(vii) $\dfrac{d}{dx}[\ln(\ln x)] = \dfrac{1}{\ln x}\dfrac{d}{dx}\ln x = \dfrac{1}{x\ln x}$

$(viii)$ $\ln\left(e^{1/x}\right) = \dfrac{1}{x} \Rightarrow \dfrac{d}{dx}\ln\left(e^{1/x}\right) = \dfrac{-1}{x^2}$.

(ix) Let $y=\left(\dfrac{1-x}{1+x}\right)^{1/2}$ then $\ln y = \frac{1}{2}\ln(1-x)-\frac{1}{2}\ln(1+x)$ and so

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2(1+x)} - \frac{1}{2(1+x)} = \frac{-(1+x)-(1-x)}{2(1-x)(1+x)} = \frac{-1}{1-x^2}.$$

Hence $\dfrac{dy}{dx} = \dfrac{(1-x)^{1/2}}{(1+x)^{1/2}}\dfrac{-1}{(1-x)(1+x)} = \dfrac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$

5. $y=x^2+2x+\cosh^{-1}(x^2+1)$

$$\frac{dy}{dx} = 2x+2+\frac{2x}{\sqrt{(x^2+1)^2-1}} = 2x+2+\frac{2}{\sqrt{x^2+2}}$$

$x=2+\sin t \Rightarrow \dfrac{dx}{dt} = \cos t = -1$ when $t=\pi$.

Also, when $t=\pi$, $x=2$ and so $\dfrac{dy}{dx} = 4+2+\dfrac{2}{\sqrt{6}} = 6.8165$

Hence $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt} = -6.8165$

6. $y = x\sin x + \ln x \;\Rightarrow\; \dfrac{dy}{dx} = \sin x + x\cos x + \dfrac{1}{x} = \dfrac{(x\sin x + x^2\cos x + 1)}{x}$

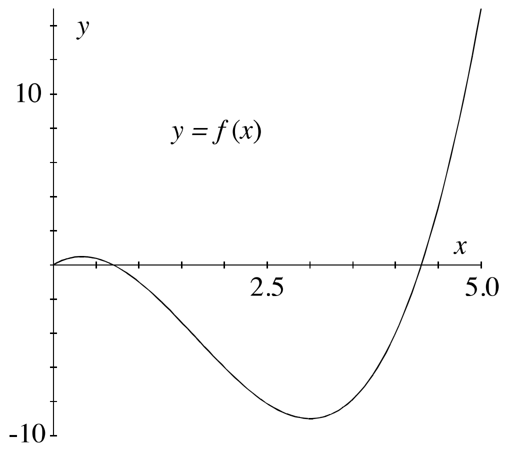
Hence $\dfrac{dx}{dy} = \dfrac{x}{x\sin x + x^2\cos x + 1}$ and so

$$\begin{aligned}\frac{d^2x}{dy^2} &= \frac{d}{dx}\left(\frac{x}{x\sin x + x^2\cos x + 1}\right)\frac{dx}{dy} \\ &= \frac{[(x\sin x + x^2\cos x + 1) - x(\sin x + 3x\cos x - x^2\sin x)]}{(x\sin x + x^2\cos x + 1)^2}\left(\frac{dx}{dy}\right) \\ &= \frac{(-2x^2\cos x + x^3\sin x + 1)x}{(x\sin x + x^2\cos x + 1)^3}\end{aligned}$$

7. If $\dfrac{dy}{dx} = P(x)$ then $\dfrac{d^2x}{dy^2} = \dfrac{d}{dy}\left(\dfrac{dx}{dy}\right) = \dfrac{d}{dy}\left(\dfrac{1}{P(x)}\right) = \dfrac{d}{dx}\left(\dfrac{1}{P(x)}\right)\dfrac{dx}{dy} = \dfrac{-P'}{P^2}\dfrac{1}{P} = -\dfrac{P'}{P^3}$

and so $\dfrac{d^2x}{dy^2} = -\dfrac{d^2y}{dx^2}\Big/\left(\dfrac{dy}{dx}\right)^3$.

8.



If $f(x) = x^3 - 5x^2 + 3x$ then $f'(x) = 3x^2 - 10x + 3 = (3x-1)(x-3)$ which is zero when $x = 1/3$ and $x = 3$. Thus the largest (and smallest) value of $f(x)$ occurs at one of $x = 0, 1/3, 3, 5$ (because x is restricted to the range $0 \le x \le 5$). Now $f(0) = 0,\; f(1/3) = 13/27,$ $f(3) = -9,\; f(5) = 15$ and so the largest value (in the given range) is 15 and the smallest value is -9.

9. If $P = \dfrac{V^2R}{(r+R)^2}$ (where V and r are constants) then

$\dfrac{dP}{dR} = V^2\dfrac{(r+R)^2 \times 1 - R \times 2(r+R)}{(r+R)^4} = \dfrac{V^2}{(r+R)^3}(r + R - 2R) = \dfrac{V^2(r-R)}{(r+R)^3}.$

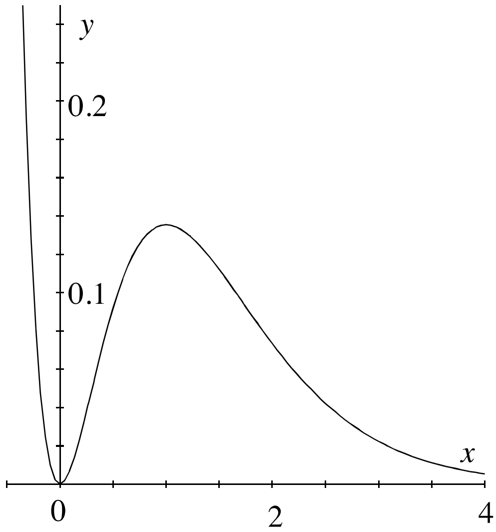
Hence $\dfrac{dP}{dR} = 0$ when $R = r$. Also $\dfrac{d^2P}{dR^2} = V^2\left[-3(r+R)^{-4}(r-R) - \dfrac{1}{(r+R)^3}\right] < 0$ when $R = r$. Thus P has a minimum at $R = r$.

10. If $y = x^2\exp(-2x)$ then

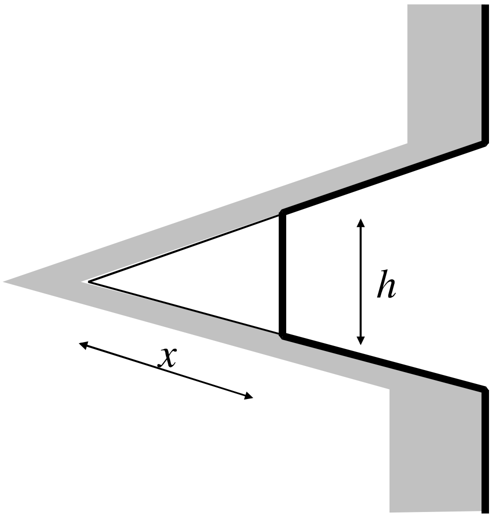
$\dfrac{dy}{dx} = 2xe^{-2x} + x^2e^{-2x}(-2) = 2e^{-2x}(x - x^2)$ and

$\dfrac{d^2y}{dx^2} = 2e^{-2x}[(1-2x) + (x-x^2)(-2)] = 2e^{-2x}(1 - 4x + 2x^2).$

Thus $\frac{dy}{dx} = 0$ when $x = x^2 \Rightarrow x = 0$ or 1.
Also, $x = 0 \Rightarrow \frac{d^2y}{dx^2} > 0 \Rightarrow$ minimum and $x = 1 \Rightarrow \frac{d^2y}{dx^2} < 0 \Rightarrow$ maximum.
Furthermore $\frac{d^2y}{dx^2} = 0$ when $2x^2 - 4x + 1 = 0$ $\Rightarrow x = 1 \pm 1/\sqrt{2}$ which are the points of inflexion.

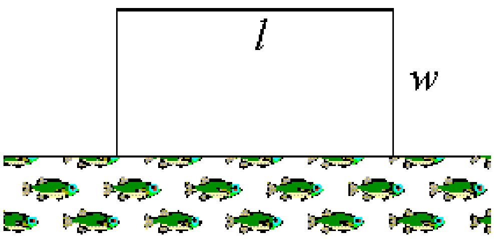


11.

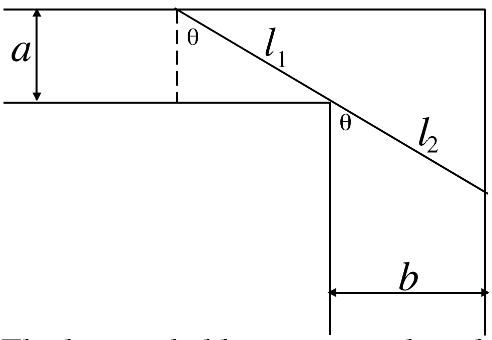


By similar triangles $\frac{x}{h} = \frac{10}{1} \Rightarrow h = x/10$.
Let C be the cost of the project, then
$C = 2(10 - x)10^4 + h^2 10^6 = 10^4(20 - 2x + x^2)$
$\Rightarrow \frac{dC}{dx} = 10^4(-2 + 2x)$ and $\frac{d^2C}{dx^2} = 2 \times 10^4 > 0$.
Thus the cost is a minimum when $x = 1$ and this minimum cost is $2 \times 9 \times 10^4 + 10^4 = 19 \times 10^4$ pounds.

12.



Let l denote the length of the field and w its width. Then the area is $A = lw \Rightarrow w = A/l$ and A is fixed. The length of fencing is
$L = l + 2w = l + 2A/l \Rightarrow \frac{dL}{dl} = 1 - \frac{2A}{l^2}$ and $\frac{d^2L}{dl^2} = 4A/l^3 > 0$. Thus the minimum length of fencing occurs when $l^2 = 2A$ which implies that $w = A/l = l/2$. So the length is twice the width.

13. 

With θ as shown, the length l of the line is given by $l = l_1 + l_2 = \dfrac{a}{\cos\theta} + \dfrac{b}{\sin\theta}$

$\Rightarrow \dfrac{dl}{d\theta} = \dfrac{a\sin\theta}{\cos^2\theta} - \dfrac{b\cos\theta}{\sin^2\theta}$. This is zero when $a\sin^3\theta = b\cos^3\theta \Rightarrow \tan\theta = \left(\dfrac{b}{a}\right)^{1/3}$.

The longest ladder correspond to the minimum value of l which occurs when $\tan\theta = \left(\dfrac{b}{a}\right)^{1/3} \Rightarrow \cos\theta = (1+\tan^2\theta)^{-1/2} = \left[\dfrac{a^{2/3}}{a^{2/3}+b^{2/3}}\right]^{1/2}$ and $\sin\theta = \left[\dfrac{b^{2/3}}{a^{2/3}+b^{2/3}}\right]^{1/2}$.

Hence the longest ladder has length $l = \left(a^{2/3}+b^{2/3}\right)^{1/2}(a/a^{1/3}+b/b^{1/3}) = (a^{2/3}+b^{2/3})^{3/2}$

14. $f(x) = (1+x^2)\tan^{-1}x \Rightarrow f'(x) = (1+x^2)\dfrac{1}{(1+x^2)} + 2x\tan^{-1}x$ and so $f'(0) = 1$.

$f(x) = \exp[x\ln\{\ln(3x+2)\}] \Rightarrow f'(x) = f(x)\dfrac{d}{dx}[x\ln\{\ln(3x+2)\}]$ and so

$f'(x) = f(x)\left[x\dfrac{d}{dx}\ln\{\ln(3x+2)\} + \ln\{\ln(3x+2)\}\right]$

$\Rightarrow f'(0) = f(0)\ln(\ln 2) = \ln(\ln 2)$.

In general, if $f(x) = \exp[xh(x)]$ then $f'(0) = h(0)$.

$f(x) = \dfrac{x(1-x^2)^2}{\sqrt{1+x^2}} = xh(x)$, say. So $f'(x) = xh'(x) + h(x) \Rightarrow f'(0) = h(0) = 1$.