

Examples 1: Functions of Real Variables

1. Sketch the graphs of the following functions. You should sketch the graph in such a way that it is clear how to extend your picture and mark on all important points (including crossings, turning points etc). Remember to label both of the axes and the curve itself. Further, find the domain and the range of each function.

(i) $y = \sin x$	(ii) $y = \sin(2x)$.	(iii) $y = \sin(x + \frac{\pi}{2})$.
(iv) $y = \sin x + 3$.	(v) $y = 3 \sin x$.	(vi) $y = 3 \sin(2x)$.
(vii) $y = \cos x$	(viii) $y = -\cos x$.	(ix) $y = \cos(x + \pi)$.
(x) $y = \cos(x + \frac{\pi}{2}) - 1$.	(xi) $y = \cos(3x)$.	(xii) $y = \cos(\frac{1}{2}x)$.
(xiii) $y = x $.	(xiv) $y = x - 1 $.	(xv) $y = x + 2$.
(xvi) $y = 2 x $.	(xvii) $y = x^2 + 3x - 4$.	(xviii) $y = -x^2 - 4$.
(xix) $y = 2x^2 + 4x - 5$.	(xx) $y = x^4$.	

2. For each of the following functions, state whether it is even, odd or neither:

(i) $\sin x$ (ii) $\cos x$ (iii) x^3 (iv) $x^3 + 1$ (v) $\ln|x|$

3. Examine the following functions and decide whether they are continuous or discontinuous. State the range and domain of each of the functions.

(i) $f(x) = \begin{cases} 1 + x^2 & (x \neq 1) \\ 5 & (x = 1) \end{cases}$

(ii) $f(x) = |x|$

(iii) $f(x) = \operatorname{sgn} x = \begin{cases} 1 & (x > 0) \\ 0 & (x = 0) \\ -1 & (x < 0) \end{cases}$ [This is the signum function.]

4. For each of the following state the domain of f , find $f^{-1}(x)$ and give the domain of f^{-1} .

(i) $f(x) = \frac{3x+2}{3-x} + 2$. (ii) $f(x) = \frac{3x-4}{2-3x} - 2$.

(iii) $f(x) = \frac{2}{3-x} + 1$. (iv) $f(x) = \frac{3x+2}{3-2x} + \frac{3}{5}$.

5. Prove the following identities from the basic definitions of the hyperbolic functions:

(i) $\operatorname{sech}^2 x + \tanh^2 x = 1$

(ii) $\sinh 2x = 2 \sinh x \cosh x$

$$(iii) \tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$(iv) \cosh 3x = 4 \cosh^3 x - 3 \cosh x$$

6. Use the Binomial theorem to expand the following:

$$(i) (a - b)^3, \quad (ii) (2a + 3b)^3, \quad (iii) (1 + 2x)^5, \quad (iv) (1 - x)^6.$$

7. Find the first four terms in the binomial expansion of the function

$$f(x) = (1 - x)^{1/2}, \quad |x| < 1.$$

Let $g(x)$ be the function formed by taking the first two terms in the binomial expansion of $f(x)$, and let $h(x)$ be the function formed by taking the first three terms of the expansion. If $x = 9/25$, what are the percentage errors that result from using $g(x)$ and $h(x)$ as approximations for $f(x)$? [Do not use a calculator].

8. Find the first four terms in the binomial expansion of the function

$$f(x) = (1 + 3x)^{2/3}.$$

For what range of values of x would you expect the binomial series to converge?

Answers

1. Use <http://www.wolframalpha.com>, for example.

2. (i) odd, (ii) even, (iii) odd, (iv) neither. (v) even

3. (i) discontinuous, (ii) continuous, (iii) discontinuous.

4. (i) Domain of f is all real numbers apart from 3. $f^{-1}(x) = \frac{3x - 8}{x + 1}$, domain = \mathbb{R} .

(ii) Domain of f is all real numbers apart from $\frac{2}{3}$. $f^{-1}(x) = \frac{2x + 4}{3x + 3}$, domain = \mathbb{R} .

(iii) Domain of f is all real numbers apart from 3. $f^{-1}(x) = \frac{3x - 5}{x - 1}$, domain = \mathbb{R} .

(iv) Domain of f is all real numbers apart from $\frac{2}{3}$. $f^{-1}(x) = \frac{15x - 19}{10x + 9}$, domain = \mathbb{R} .

6. (i) $a^3 - 3a^2b + 3ab^2 - b^3$, (ii) $8a^3 + 36a^2b + 54ab^2 + 27b^3$,

(iii) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$,

(iv) $1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

7. $1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots$, 2.5% error ($g(x)$) and 0.475% error ($h(x)$).

8. $1 + 2x - x^2 + \frac{4}{3}x^3 + \dots$, expect convergence for $|x| < \frac{1}{3}$.