Examples 1: Functions of Real Variables

Solutions

Solutions for Q1 can be obtained using any graphing software, such as that accessible on Wolfram Alpha (http://www.wolframalpha.com).

2) (i) odd : $\sin(-x) = -\sin x$

(ii) even : $\cos(-x) = \cos x$

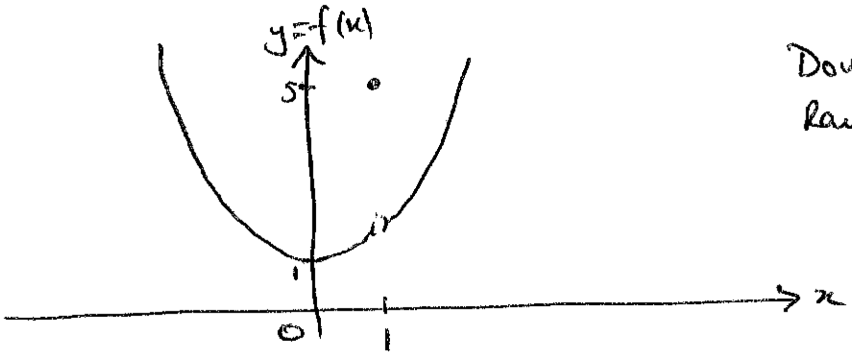
(iii) odd : $(-x)^3 = -x^3$

(iv) neither : $(-x)^3 + 1 = -x^3 + 1$

(v) even : $\ln|-x| = \ln|x|$.

3) (i) As $x \to 1$, from either above or below, $f(x) \to 2$

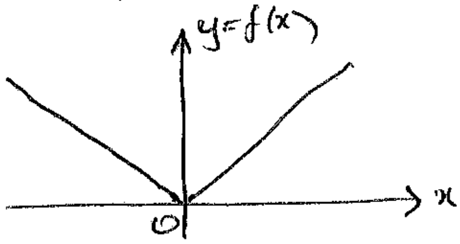
So $f(1) = 5 \Rightarrow f(x)$ is <u>discontinuous</u> at $x = 1$



Domain: $-\infty < x < \infty$

Range : $1 \leq f(x) < \infty$

(ii) $f(x) = |x|$ is continuous because $|x| \to 0$ as $x \to 0$ either from above or below



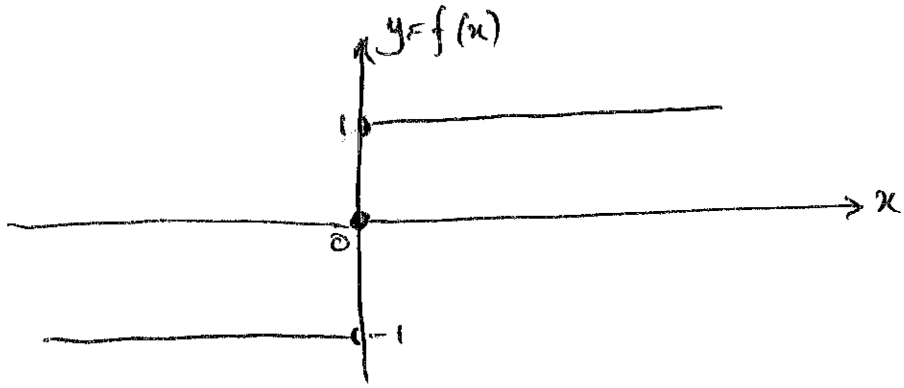
Domain: $-\infty < x < \infty$

Range : $0 \leq x < \infty$

(iii) $f(x) \Rightarrow 1$ when x is just greater than 0

$f(x) = -1$ when x is just less than 0

$f(0) = 0$ $\therefore$ <u>discontinuous</u>.



4. (i) $f(x) = \dfrac{3x+2}{3-x} + 2$ is defined for all $x \neq 3$.

$\therefore$ The best choice of domain is $x \in \mathbb{R}, x \neq 3$.

Let $y = \dfrac{3x+2}{3-x} + 2$.

Then $y - 2 = \dfrac{3x+2}{3-x}$

so $(3-x)(y-2) = 3x+2$

so $3y - 6 - xy + 2x = 3x + 2$

so $x + xy = 3y - 8$

so $x(1+y) = 3y - 8$

so $x = \dfrac{3y-8}{1+y}$ (for $y \neq -1$).

Thus $f^{-1}(x) = \dfrac{3x-8}{1+x}$ for $x \neq -1$.

That is, the domain of $f^{-1}(x)$ is $x \in \mathbb{R}, x \neq -1$.

(ii) $f(x) = \dfrac{3x-4}{2-3x} - 2$ is defined for all $x \neq 2/3$.

$\therefore$ The best choice of domain is $x \in \mathbb{R}, x \neq 2/3$.

Let $y = \dfrac{3x-4}{2-3x} - 2$. Then $(y+2)(2-3x) = 3x - 4$

so $2y - 3xy + 4 - 6x = 3x - 4$

so $9x + 3xy = 2y + 8$

so $x(9 + 3y) = 2(y+4)$

so $x = \dfrac{2(y+4)}{3(y+3)}$ for $y \neq -3$.

$\therefore \quad f^{-1}(x) = \dfrac{2(x+4)}{3(x+3)}$ with domain $x \in \mathbb{R}, x \neq -3$.

(iii) $f(x) = \dfrac{2}{3-x} + 1$ has domain $x \in \mathbb{R}, x \neq 3$.

Let $y = \dfrac{2}{3-x} + 1$. Then $(3-x)(y-1) = 2$

so $3y - 3 - xy + x = 2$

so $x(1-y) = 5 - 3y$

so $x = \dfrac{5-3y}{1-y}$ for $y \neq 1$.

$\therefore \quad f^{-1}(x) = \dfrac{5-3x}{1-x}$ with domain $x \in \mathbb{R}, x \neq 1$.

(iv) $f(x) = \dfrac{3x+2}{3-2x} + \dfrac{3}{5}$ has domain $x \in \mathbb{R}, x \neq 3/2$.

Let $y = \dfrac{3x+2}{3-2x} + \dfrac{3}{5}$. Then $5y - 3 = \dfrac{15x+10}{3-2x}$

so $(5y-3)(3-2x) = 15x+10$

so $15y - 10xy - 9 + 6x = 15x + 10$

so $9x + 10xy = 15y - 19$

so $x(9+10y) = 15y - 19$

so $x = \dfrac{15y-19}{9+10y}$ for $y \neq -9/10$.

$\therefore \quad f^{-1}(x) = \dfrac{15x-19}{9+10x}$ with domain $x \in \mathbb{R}, x \neq -9/10$

5) (i) $\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$$\operatorname{sech}^2 x + \tanh^2 x = \frac{1}{\cosh^2 x} + \frac{\sinh^2 x}{\cosh^2 x}$$

$$= \frac{1 + \sinh^2 x}{\cosh^2 x}$$

Now, $1 + \sinh^2 x = 1 + \frac{1}{4}(e^x - e^{-x})^2$

$$= 1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$= \frac{1}{4}(e^{2x} + 2 + e^{-2x})$$

$$= \cosh^2 x.$$

$\therefore \operatorname{sech}^2 x + \tanh^2 x = \frac{\cosh^2 x}{\cosh^2 x} = 1$

(ii) $2 \sinh x \cosh x = \frac{1}{2}(e^x - e^{-x})(e^x + e^{-x})$

$$= \frac{1}{2}(e^{2x} - 1 + 1 - e^{-2x})$$

$$= \frac{1}{2}(e^{2x} - e^{-2x})$$

$$= \underline{\sinh 2x}.$$

(iii) $\cosh(x+y) = \frac{1}{2}(e^{x+y} + e^{-x-y})$

$\sinh(x+y) = \frac{1}{2}(e^{x+y} - e^{-x-y})$

$$\sinh x \cosh y = \frac{1}{4}(e^{x}-e^{-x})(e^{y}+e^{-y})$$
$$= \frac{1}{4}(e^{x+y}-e^{-x+y}+e^{x-y}-e^{-x-y})$$

$$\cosh x \sinh y = \frac{1}{4}(e^{x}+e^{-x})(e^{y}-e^{-y})$$
$$= \frac{1}{4}(e^{x+y}+e^{-x+y}-e^{x-y}-e^{-x-y})$$

$$\therefore \sinh x \cosh y + \cosh x \sinh y = \frac{1}{2}(e^{x+y}-e^{-x-y})$$
$$= \sinh(x+y)$$

$$\cosh x \cosh y = \frac{1}{4}(e^{x}+e^{-x})(e^{y}+e^{-y})$$
$$= \frac{1}{4}(e^{x+y}+e^{-x+y}+e^{x-y}+e^{-x-y})$$

$$\sinh x \sinh y = \frac{1}{4}(e^{x}-e^{-x})(e^{y}-e^{-y})$$
$$= \frac{1}{4}(e^{x+y}-e^{-x+y}-e^{x-y}+e^{-x-y})$$

$$\therefore \cosh x \cosh y + \sinh x \sinh y = \frac{1}{2}(e^{x+y}+e^{-x-y})$$
$$= \cosh(x+y).$$

$$\therefore \tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$$
$$= \frac{\tanh x \cosh y + \sinh y}{\cosh y + \tanh x \sinh y}$$
$$= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.$$

5) (iv) $\cosh 2x = \cosh^2 x + \sinh^2 x$

$= 2\cosh^2 x - 1$

$\cosh 3x = \cosh(2x + x)$

$= \cosh 2x . \cosh x + \sinh 2x . \sinh x$

$= (2\cosh^2 x - 1)\cosh x + 2\cosh x \sinh x . \sinh x$

$= (2\cosh^2 x - 1)\cosh x + 2\cosh x(\cosh^2 x - 1)$

$= 2\cosh^3 x - \cosh x + 2\cosh^3 x - 2\cosh x$

$= \underline{4\cosh^3 x - 3\cosh x}$.

6) (i) $(a-b)^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$

$= \underline{a^3 - 3a^2 b + 3ab^2 - b^3}$

(ii) $(2a+3b)^3 = (2a)^3 + 3(2a)^2 3b + 3.2a.(3b)^2 + (3b)^3$

$= \underline{8a^3 + 36a^2 b + 54ab^2 + 27b^3}$

(iii) $(1+2x)^5 = 1 + {}^5C_1(2x) + {}^5C_2(2x)^2 + {}^5C_3(2x)^3 + {}^5C_4(2x)^4$

$+ {}^5C_5(2x)^5$.

${}^5C_1 = \frac{5!}{1!4!} = 5$; ${}^5C_2 = \frac{5!}{2!3!} = \frac{5.4}{2} = 10$.

${}^5C_3 = \frac{5!}{3!2!} = 10$; ${}^5C_4 = \frac{5!}{4!1!} = 5$; ${}^5C_5 = 1$

$\underline{(1+2x)^5 = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5}$

6) (iv) $(1-x)^6 = \sum_{r=0}^{6} {}^6C_r (-x)^r$

${}^6C_0 = 1$; ${}^6C_1 = 6$; ${}^6C_2 = \frac{6!}{2!\,4!} = \frac{6.5}{2} = 15$; ${}^6C_3 = \frac{6!}{3!\,3!} = \frac{6.5.4}{6.} = 20$

${}^6C_4 = \frac{6!}{4!\,2!} = 15$; ${}^6C_5 = \frac{6!}{5!\,1!} = 6$; ${}^6C_6 = 1$

$\therefore (1-x)^6 = 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$.

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7)

$f(x) = (1-x)^{1/2}$

$= 1 + \frac{1}{2}(-x) + \frac{1}{2!}\frac{1}{2}\left(\frac{1}{2}-1\right)(-x)^2 + \frac{1}{3!}\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(-x)^3 + \cdots$

$= 1 - \frac{1}{2}x + \frac{1}{2}.\frac{1}{2}.\left(-\frac{1}{2}\right).x^2 + \frac{1}{6}.\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right).(-x)^3 + \cdots$

$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \cdots$

$\therefore g(x) = 1 - \frac{1}{2}x$

$h(x) = 1 - \frac{1}{2}x - \frac{1}{8}x^2$

If $x = 9/25$, $f(x) = \left(1 - \frac{9}{25}\right)^{1/2} = \left(\frac{25-9}{25}\right)^{1/2} = \left(\frac{16}{25}\right)^{1/2}$

$= \frac{4}{5}$.

$\therefore \underline{f(9/25) = 4/5}$.

$g(9/25) = 1 - \frac{9}{50} = \frac{50-9}{50} = \underline{\frac{41}{50}}$

$h(9/25) = \frac{41}{50} - \frac{1}{8}\cdot\left(\frac{9}{25}\right)^2 = \frac{41}{50} - \frac{81}{8.625}$

$= \frac{41}{50} - \frac{81}{5000}$

$= \frac{4100-81}{5000}$

$= \underline{\frac{4019}{5000}}$.

$f(9/25) = \frac{4000}{5000}$

$g(9/25) = \frac{4100}{5000}$

$h(9/25) = \frac{4019}{5000}$.

$\frac{g(9/25) - f(9/25)}{f(9/25)} = \frac{100}{4000} = 0.0.25$ (2.5% error)

$\frac{h(9/25) - f(9/25)}{f(9/25)} = \frac{19}{4000} = \frac{19}{40}\%$ (0.475 % error)

[$\therefore$ gives a good approximation to $f(x)$].

8) $f(x) = (1+3x)^{2/3} = (1+y)^{2/3} \quad (y = 3x)$

$$(1+y)^{2/3} = 1 + \frac{2}{3}y + \frac{1}{2!}\frac{2}{3}\left(\frac{2}{3}-1\right)y^2 + \frac{1}{3!}\frac{2}{3}\cdot\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)y^3 + \cdots$$

$$= 1 + \frac{2}{3}y + \frac{1}{3}\left(-\frac{1}{3}\right)y^2 + \frac{1}{9}\cdot\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)y^3 + \cdots$$

$$= 1 + \frac{2}{3}y - \frac{1}{9}y^2 + \frac{4}{81}y^3 + \cdots$$

and $y = 3x$
$y^2 = 9x^2$
$y^3 = 27x^3$

$$\therefore (1+3x)^{2/3} = 1 + \frac{2}{3}\cdot 3x - \frac{1}{9}\cdot 9x^2 + \frac{4}{81}\cdot 27x^3 + \cdots$$

$$= 1 + 2x - x^2 + \frac{4}{3}x^3 + \cdots$$

Since $(1+x)^p$ converges for $|x| < 1$, expect convergence for $|3x| < 1$ ie $|x| < \frac{1}{3}$.