The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2016–2017**

MAS140 Mathematics (Chemical) **3 hours**

Attempt ***ALL*** *the questions.*
Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.
All solutions should be justified in full. Calculators should be relied upon for simple steps like basic arithmetic and plugging numbers into elementary functions.

Section A

A1 Let $f(x) = \dfrac{x}{x^2+1}$. Find the stationary points and sketch a graph of $y = f(x)$.

A2 Let $f(x) = \dfrac{\ln x}{x}$. State the domain and the range of $y = f(x)$.

A3 Let $f(x,y) = e^x \sin y$. Find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$.

A4 Using l'Hospital's Rule, or otherwise, evaluate $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$.

A5 Find all complex numbers z for which $|z - i| = |z + i|$.

A6 Let $\boldsymbol{a} = (1,1,1)$ and $\boldsymbol{b} = (1,-2\lambda,1)$. Find a value of λ such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular. Also, find a value of λ such that $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.

A7 Find the definite integral $\int_0^1 xe^{-x}\,dx$.

A8 Find the indefinite integral $\int x \ln x\,dx$.

A9 For $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, find A^{-1}. Without doing any further matrix multiplication, indicate why $AA^T = A^T A = I$ holds.

A10 Consider the following system of equations

$$\begin{cases} x + 2y = 0, \\ \lambda x + 3y = 0, \end{cases}$$

and let $A = \begin{pmatrix} 1 & 2 \\ \lambda & 3 \end{pmatrix}$. Considering the determinant of A, find the value of λ for the above system to have a unique solution. Also, find the value of λ for the system to have infinitely many solutions.

A11 Find the general solution of the differential equation

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}.$$

A12 Find the general solution of the differential equation

$$\frac{dy}{dx} + y = x.$$

Section B

B1 (i) Let $g(y) = \sin^{-1} y + y\sqrt{1-y^2}$. Show that $\dfrac{dg}{dy} = 2\sqrt{1-y^2}$.

(ii) Consider the function $f(x) = g\left(\dfrac{x}{a}\right)$ where a is a fixed constant and g is the function introduced in (i). Calculate $\dfrac{df}{dx}$ to show that

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C,$$

where C is constants.

B2 Find all the cube roots of $z = \sqrt{3} + i$ and plot them on the Argand diagram.

B3 The position vector $\boldsymbol{r}(t)$ of a particle is given by

$$\boldsymbol{r}(t) = (t^2+t, t^3-t, t^3-t^2).$$

(i) When $t = 1$, find the velocity $\boldsymbol{v} = \dfrac{d\boldsymbol{r}}{dt}$ and acceleration $\boldsymbol{a} = \dfrac{d^2\boldsymbol{r}}{dt^2}$.

(ii) Again, when $t = 1$, find a unit vector $\boldsymbol{t}$ in the direction of $\boldsymbol{v}$ and the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{v}$.

B4 Consider the differential equation

$$\frac{d^2y}{dx^2} - (a+1)\frac{dy}{dx} + ay = 0.$$

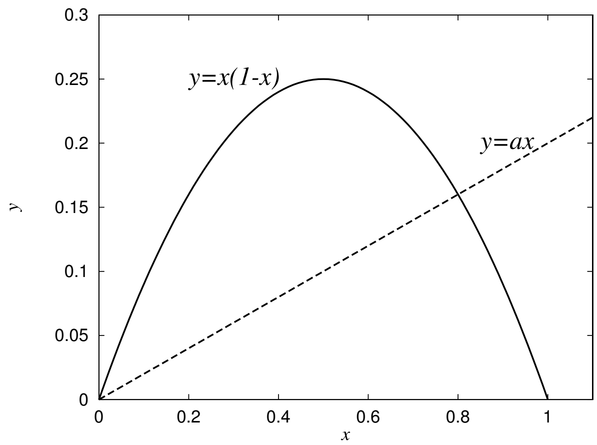
(i) Find the general solution when $a \neq 1$.

(ii) Find the general solution when $a = 1$.

(iii) Using l'Hôpital rule, find

$$\lim_{a\to 1} \frac{e^{ax}-e^x}{a-1},$$

thereby relating the solutions in (i) and (ii).



B5 (i) Show that

$$\int_0^{\alpha} x(\alpha - x)\, dx = \frac{\alpha^3}{6},$$

for $\alpha > 0$.

(ii) Consider an area bounded by a graph of $y = x(1-x)$ and the x-axis in the above figure. Find a constant a $(0 < a < 1)$ such that $y = ax$ bisects the area.

B6 Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 1 \\ 2 & -4 & 0 \end{pmatrix}$. Find all the eigenvalues of A and corresponding eigenvectors.

B7 Show that

$$\int_0^{\pi/2} \ln \sin x\, dx = -\frac{\pi}{2} \ln 2$$

by making use of either

Hint (1) $$\int_0^{\pi/2} \ln \sin x\, dx = \int_0^{\pi/2} \ln \cos x\, dx = \frac{1}{2} \int_0^{\pi/2} \ln \frac{\sin 2x}{2}\, dx,$$

or

Hint (2) $$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \ldots \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}} \quad (n \geq 2).$$

There is no need to prove the Hints.

B8 Using Gauss elimination, solve the following system of linear equations

$$\begin{cases} x + 2y + 7z = 3, \\ -2x + 5y + 4z = 3, \\ -5x + 6y - 3z = 1. \end{cases}$$

End of Question Paper