Data provided: Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

MAS152 Essential Mathematical Skills and Techniques **3 hours**

Attempt ***ALL*** *questions.*

Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.

All solutions should be justified in full. Calculators should be relied upon only for simple steps like basic arithmetic and plugging numbers into elementary functions.

Section A

A1 Let $f(x) = \dfrac{x}{x+4}$. Sketch the curve $y = f(x)$.

A2 Let $f(x) = e^{2x} - 1$. Find $f^{-1}(x)$ and state its domain and range.

A3 If $f(x, y) = 4x^2\sqrt{y} + 5\cos(xy)$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

A4 Evaluate $\lim_{x\to 0}\left(\dfrac{x\tanh x}{\sin 2x}\right)$ using l'Hôpital's Rule.

A5 Find all the complex numbers z for which $|z - 1 - i| = 1$ and $\mathrm{Re}(z) = \mathrm{Im}(z)$.

A6 If $\mathbf{a} = (7, -2, -5)$ and $\mathbf{b} = (5, 1, 3)$, evaluate $\mathbf{a} \cdot \mathbf{b}$. Find a non-zero vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

A7 Find the definite integral $\displaystyle\int_0^{\pi} (x + 1) \sin \frac{x}{2}\, dx$.

A8 Find the indefinite integral $\displaystyle\int x(\cos x^2)^2\, dx$.

A9 Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Find A^T and B^T and hence show that $(AB)^T = B^T A^T$.

A10 Find the general solution of the differential equation

$$x\frac{dy}{dx} = \frac{e^x}{x^2} - 3y.$$

A11 Find A^{-1} for the matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$. Use this to solve the simultaneous equations

$$\begin{aligned} 2x + 4y &= 14 \\ x - 3y &= -8. \end{aligned}$$

A12 Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$$

such that $y = 0$ when $x = 0$.

Section B

B1 Find all the stationary points of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 - 1$ and show that the stationary points not at $(0, 0)$ are minima.

B2 By evaluating all the necessary derivatives of $y = \ln(1 + x^2)$, find the first 2 non-zero terms of the Maclaurin Series expansion of y. Show that this series can also be obtained from the Maclaurin Series of $y = \ln(1+x)$ given on the Formula Sheet.

B3 Find the modulus and principal argument of the complex numbers $z_1 = 1 + i$ and $z_2 = \sqrt{3} + i$. Hence find all complex numbers z that satisfy the equation $z^6 = \dfrac{z_1}{z_2}$ and plot them on an Argand diagram.

B4 The position vector of a particle, $\mathbf{r}(t)$, is given by

$$\mathbf{r}(t) = (2t^2, t^2 - 4t, 3t - 5).$$

(i) Find the velocity and acceleration vectors of the particle.

(ii) Find the unit vector in the direction of $\mathbf{r}$ at $t = 1$. Hence show that, in this direction, the component of the velocity is 4 times the component of the acceleration.

B5 Evaluate the definite integral

$$\int_1^2 \frac{2t^2 + 3t + 1}{t^3 + t}\, dt$$

writing your answer to 2 decimal places.

B6 Find the value of α for which the following system of equations has infinitely many solutions and then find those solutions.

$$\begin{aligned} 4x - y - z &= 2 \\ 2x + \alpha y + z &= 4 \\ x - 2y - 2z &= -3 \end{aligned}$$

For the case $\alpha = -2$, without solving the equations state how many solutions you would expect and write down the solution of the corresponding homogeneous system of equations.

B7 Let $A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$. Find all eigenvalues and eigenvectors of A.

B8 Using Laplace Transforms or otherwise, find the solution to the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$$

subject to the initial conditions $y = 0$ and $\dfrac{dy}{dt} = 0$ at $t = 0$.

End of Question Paper