Data provided: Formula Sheet

MAS152

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

MAS152 Essential Mathematical Skills and Techniques **3 hours**

*Attempt **ALL** questions.*

Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.

Section A

A1 Let $f(x) = \dfrac{x+2}{x-3}$. Sketch the curve $y = f(x)$.

A2 Let $f(x) = \dfrac{1}{2}e^{x^3+1}$. Find $f^{-1}(x)$ and state its domain and range.

A3 If $f(x, y) = x\ln(x^2 + y^2)$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial^2 f}{\partial x \partial y}$.

A4 Use l'Hôpital's Rule to evaluate $\lim\limits_{x\to 0}\left(\dfrac{x\sin x}{\sinh^2 x}\right)$.

A5 Find all the complex numbers z for which $|z-1| = \sqrt{3}$ and $z.\overline{z} = 4$, where $\overline{z}$ is the complex conjugate of z.

A6 Find the value of t for which $\mathbf{a} = (3, -1, 2)$ and $\mathbf{b} = (4, 2, t)$ are perpendicular. For this value of t, evaluate $\mathbf{a} \times \mathbf{b}$ and find a unit vector in the direction of $\mathbf{a} \times \mathbf{b}$.

A7 Find the definite integral $\int_1^e x(\ln x)^2 dx$ using integration by parts.

A8 Find the indefinite integral $\int \frac{1}{\sqrt{4x - x^2}} dx$.

A9 Let $A = \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}$. For each of $A^{-1}B$ and $B^{-1}A$, either calculate it or say why it doesn't exist.

A10 Find the particular solution of the differential equation

$$(1 + x^2)\frac{dy}{dx} = 2xy + 2x$$

for which $y = 1$ when $x = 0$.

A11 For which real values of α does the system of linear equations below have infinitely many solutions for x, y and z? You do not need to find x, y and z.

$$\begin{array}{rcrcrcl} 2x & + & \alpha y & & & = & 0 \\ x & - & y & + & \alpha z & = & 0 \\ x & + & 3y & + & z & = & 0. \end{array}$$

A12 Find the values of k for which $y = e^{kx}$ is a solution to

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0.$$

Section B

B1 Find the Maclaurin expansion of $f(x) = \sin^{-1} x$ up to and including the term involving x^3. By considering the expansion of $(1+y)^{\frac{1}{2}}$ or otherwise, find the Maclaurin expansion of $(1-\sin^{-1} x)^{\frac{1}{2}}$ up to and including the x^3-term.

B2 Find and classify the stationary points of $f(x,y) = x^3 + 4xy - 2y^2 - 7x$.

B3 (i) Let $z = \cos\theta + i\sin\theta$. By calculating z^5 in two different ways, show that $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(ii) Use part (i) to show that $\cos\left(\frac{3\pi}{10}\right)$ is a solution to $16x^4 - 20x^2 + 5 = 0$, and hence find its precise value (expressed using square-roots, not as a decimal). (You may use without justification that $\cos\left(\frac{3\pi}{10}\right)$ is the smallest positive root of this equation.)

B4 Particles A and B have position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ at time t given by

$$\begin{aligned}\mathbf{r}_A &= (\cos(\pi t), \sin(\pi t), t)\\ \mathbf{r}_B &= (1-2t, 0, t^2).\end{aligned}$$

(i) Show that at $t = 0$, particles A and B are in the same position and that the velocity of A is perpendicular to the velocity of B.

(ii) Determine whether the particles collide at some value of $t > 0$.

(iii) Describe the path taken by particle A assuming its path is unaffected by any collision with particle B, and draw a rough sketch.

B5 Let $t = \tanh(x)$. Show that $\cosh(2x) = (1+t^2)/(1-t^2)$, $\sinh(2x) = 2t/(1-t^2)$ and $\frac{dx}{dt} = \frac{1}{1-t^2}$. Hence show that

$$\int \frac{dx}{\sinh(2x)} = \frac{1}{2}\ln|\tanh x| + c$$

and find $\int \frac{dx}{\cosh(2x)}$.

(Note: there are formulas for hyperbolic functions on the formula sheet.)

B6 Recall that a 2×2 matrix A represents a transformation of the plane: given a point with coordinates (x, y), the new coordinates are given by $A \begin{pmatrix} x \\ y \end{pmatrix}$.

By working out their effects on the square with corners at $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$ (or otherwise), describe the transformations represented by the matrices below.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$
$$A_4 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

B7 Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$. Find all eigenvalues and eigenvectors of A.

B8 Find the general solution to the differential equation

$$y'' - 4y' + 4y = e^x \cos x.$$

End of Question Paper