NOTES ON BACKWARDS REASONING

Consider proving the identity $\operatorname{cosec}\theta - \sin\theta = \cos\theta\cot\theta$. This question is often answered with the right ideas — but incorrectly — in the following manner.

$$\begin{array}{lrcl} & \operatorname{cosec}\theta - \sin\theta & = & \cos\theta\cot\theta \\ \text{so} & \frac{1}{\sin\theta} - \sin\theta & = & \cos\theta\frac{\cos\theta}{\sin\theta} \\ \text{so} & 1 - \sin^2\theta & = & \cos^2\theta \qquad \text{(multiplying by } \sin\theta\text{)} \\ \text{so} & 1 & = & \sin^2\theta + \cos^2\theta, \end{array}$$

and the final identity always holds.

What is wrong here? Well, we have proved that *if* $\operatorname{cosec}\theta - \sin\theta = \cos\theta\cot\theta$ *then* $1 = \sin^2\theta + \cos^2\theta$. This is the wrong way around! We know that $1 = \sin^2\theta + \cos^2\theta$ and want to prove that $\operatorname{cosec}\theta - \sin\theta = \cos\theta\cot\theta$. Slack logic like this can cause serious problems, such as the following 'proof' that $-2 = 2$.

$$\begin{array}{lrcl} & -2 & = & 2 \\ \text{so} & (-2)^2 & = & 2^2 \qquad \text{(squaring both sides)} \\ \text{so} & 4 & = & 4, \end{array}$$

which is true.

Clearly such 'proofs' need to be avoided. The good news is that our original attempt *can* be made into a valid proof quite easily by turning the argument upside-down. That is, our identity can be proved by the following.

$$\begin{array}{lrcl} & 1 & = & \sin^2\theta + \cos^2\theta \\ \text{so} & 1 - \sin^2\theta & = & \cos^2\theta \\ \text{so} & \frac{1}{\sin\theta} - \sin\theta & = & \cos\theta\frac{\cos\theta}{\sin\theta} \qquad \text{(dividing by } \sin\theta\text{)} \\ \text{so} & \operatorname{cosec}\theta - \sin\theta & = & \cos\theta\cot\theta, \end{array}$$

so the claimed identity holds.

But arguably the best method is to take the left-hand side of the identity and manipulate it until it is the same as the right-hand side (using the identity $1 = \sin^2\theta + \cos^2\theta$ along the way), as shown.

$$\begin{array}{rcl} \operatorname{cosec}\theta - \sin\theta & = & \frac{1}{\sin\theta} - \sin\theta \\ & = & \frac{1-\sin^2\theta}{\sin\theta} \\ & = & \frac{\cos^2\theta}{\sin\theta} \qquad \text{(using } \sin^2\theta + \cos^2\theta = 1\text{)} \\ & = & \cos\theta\cot\theta, \end{array}$$

as claimed.