

## REVISION

**5 minute review.** Very briefly talk students through what was covered in the course. (I wouldn't waste time writing this down!)

- *Functions*: curve sketching, binomial theorem, inverse functions, exponential & logarithms, trigonometric & hyperbolic functions;
- *Differentiation*: first principles, differentiation rules, parametric & implicit differentiation, partial differentiation;
- *Series*: Maclaurin & Taylor series, l'Hôpital's rule;
- *Complex numbers*: polar & exponential forms, Argand diagram, Euler's relation, de Moivre's theorem;
- *Vectors*: scalar product, vector product;
- *Integration*: substitution, parts, definite integrals, improper integrals;
- *Matrices*: multiplication, determinants, inverses, systems of equations, eigenvectors;
- *Differential equations*: separation of variables, integrating factors, second-order methods, simultaneous DEs.

**Class warm-up.** Choose a question or two from below (e.g. 1 or 3(a) or ...). These are all taken or adapted from the 2011–12 exam, which was found hard.

**Problems.** Choose from the below.

1. **Functions.** Find the stationary points and sketch the graph of  $y = \frac{x}{1+x^2}$ .
2. **Differentiation.** If  $f(x, y) = xy^2 \cosh(x^2y)$ , find  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$ .
3. **Limits.**
  - (a) Use the binomial theorem to evaluate  $\lim_{x \rightarrow 0} \frac{\sqrt{9-2x}-3}{x}$ .
  - (b) Evaluate  $\lim_{x \rightarrow 0} \frac{x - \tan x}{x - \sin x}$ .
4. **Complex numbers.** The complex numbers  $z_1$  and  $z_2$  satisfy  $\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1) \operatorname{Re}(z_2)$ . What (if anything) can you deduce about  $z_1$  and  $z_2$ ?
5. **Vectors.** The position vector of a particle at time  $t \geq 0$  is given by

$$\mathbf{r} = (6 \sin(t^2), 6 \cos(t^2), (1 + 4t)^{3/2}).$$

Find the velocity of the particle at time  $t$  and verify that the speed of the particle varies linearly with time.

6. **Integration.** Compute the indefinite integrals

(a)  $\int \frac{3(\arctan x)^2 - 1}{x^2 + 1} dx;$

(b)  $\int \frac{\sin x + 2 \cos x}{2 \sin x + \cos x} dx.$

7. **Matrices.** Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

**Selected answers and hints.**

1. We have  $\frac{dy}{dx} = (1-x^2)/(1+x^2)^2$ , so  $\frac{dy}{dx} = 0 \iff x = \pm 1$ . Thus the stationary points are at  $(1, 0.5)$  (a maximum) and  $(-1, -0.5)$  (a minimum). The graph passes through the origin and tends to zero at  $\pm\infty$ .

2.  $\frac{\partial f}{\partial x} = y^2 \cosh(x^2 y) + 2x^2 y^3 \sinh(x^2 y)$  and  $\frac{\partial f}{\partial y} = 2xy \cosh(x^2 y) + x^3 y^2 \sinh(x^2 y)$ .

3. (a) Firstly,  $\sqrt{9-2x} = (9-2x)^{\frac{1}{2}} = 9^{\frac{1}{2}}(1-\frac{2x}{9})^{\frac{1}{2}} = 3(1+\frac{1}{2}(-\frac{2x}{9})+\dots)$ , where all further terms have a factor of  $x^2$ . Thus

$$\lim_{x \rightarrow 0} \frac{\sqrt{9-2x}-3}{x} = \lim_{x \rightarrow 0} \frac{(3-\frac{x}{3}+\dots)-3}{x} = \lim_{x \rightarrow 0} \left(-\frac{1}{3}+\dots\right) = -\frac{1}{3}.$$

(b) The limit is of the form  $\frac{0}{0}$ , hence

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x - \tan x}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{1 - \sec^2 x}{1 - \cos x} \quad (\text{by l'H\^opital's Rule}) \\ &= \lim_{x \rightarrow 0} \frac{\cos^2 x - 1}{\cos^2 x - \cos^3 x} \quad (\text{rearranging}) \\ &= \lim_{x \rightarrow 0} \frac{-2 \cos x \sin x}{-2 \cos x \sin x + 3 \cos^2 x \sin x} \quad (\text{l'H\^opital again}) \\ &= \lim_{x \rightarrow 0} \frac{-2 \cos x}{-2 \cos x + 3 \cos^2 x} \quad (\text{cancelling}) \\ &= -2. \end{aligned}$$

4. Writing  $z_1 = a+ib$ ,  $z_2 = c+id$  we have  $\text{Re}(z_1 z_2) = ac-bd$  and  $\text{Re}(z_1) \text{Re}(z_2) = ac$ . Thus  $bd = 0$ , so at least one of  $z_1$  or  $z_2$  must have zero imaginary part (i.e. is real).

5. The velocity vector is

$$\begin{aligned} \dot{\mathbf{r}} &= (12t \cos(t^2), -12t \sin(t^2), (3/2)(1+4t)^{1/2}.4) \\ &= (12t \cos(t^2), -12t \sin(t^2), 6(1+4t)^{1/2}). \end{aligned}$$

Thus the speed is given by

$$\begin{aligned} |\dot{\mathbf{r}}| &= \sqrt{144t^2 \cos^2(t^2) + 144t^2 \sin^2(t^2) + 36(1+4t)} \\ &= 6\sqrt{(2t+1)^2} \\ &= 6(2t+1), \end{aligned}$$

which varies linearly with  $t$ .

6. (a) Substituting  $u = \tan^{-1} x$ ,  $\int \frac{3(\tan^{-1} x)^2 - 1}{x^2 + 1} dx = (\tan^{-1} x)^3 - \tan^{-1} x + c$ .

(b) Using the standard substitution  $t = \tan(x/2)$  and partial fractions, we find

$$\begin{aligned} \int \frac{\sin x + 2 \cos x}{2 \sin x + \cos x} dx &= \int \frac{-4t^2 + 4t + 4}{(1+t^2)(-t^2 + 4t + 1)} dt \\ &= \frac{1}{5} \int \left( \frac{-6t}{1+t^2} + \frac{8}{1+t^2} + \frac{-6t+12}{-t^2+4t+1} \right) dt \\ &= \frac{1}{5} (-3 \ln(1+t^2) + 8 \tan^{-1} t + 3 \ln(|-t^2+4t+1|)) + c. \end{aligned}$$

7. It turns out that  $A^{-1} = A^T$ .

For more details, start a thread on the discussion board.