REVISION

5 minute review. Very briefly talk students through what was covered in the course. (I wouldn't waste time writing this down!)

- *Functions*: curve sketching, binomial theorem, inverse functions, exponential & logarithms, trigonometric & hyperbolic functions;
- *Differentiation*: first principles, differentiation rules, parametric & implicit differentiation, partial differentiation;
- *Series*: Maclaurin & Taylor series, l'Hôpital's rule;
- *Complex numbers*: polar & exponential forms, Argand diagram, Euler's relation, de Moivre's theorem;
- *Vectors*: scalar product, vector product;
- *Integration*: substitution, parts, definite integrals, improper integrals;
- *Matrices*: multiplication, determinants, inverses, systems of equations, eigenvectors;
- *Differential equations*: separation of variables, integrating factors, second-order methods, simultaneous DEs.

Class warm-up. Choose a question or two from below (e.g. 1 or 3(a) or ...). These are all taken or adapted from the 2011–12 exam, which was found hard.

Problems. Choose from the below.

1. **Functions**. Find the stationary points and sketch the graph of $y = \dfrac{x}{1+x^2}$.
2. **Differentiation**. If $f(x,y) = xy^2\cosh(x^2y)$, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
3. **Limits**.
   (a) Use the binomial theorem to evaluate $\lim_{x\to 0}\frac{\sqrt{9-2x}-3}{x}$.
   (b) Evaluate $\lim_{x\to 0}\frac{x-\tan x}{x-\sin x}$.
4. **Complex numbers**. The complex numbers z_1 and z_2 satisfy $\operatorname{Re}(z_1z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2)$. What (if anything) can you deduce about z_1 and z_2?
5. **Vectors**. The position vector of a particle at time $t \geq 0$ is given by
$$\mathbf{r} = (6\sin(t^2), 6\cos(t^2), (1+4t)^{3/2}).$$
   Find the velocity of the particle at time t and verify that the speed of the particle varies linearly with time.
6. **Integration**. Compute the indefinite integrals
   (a) $\displaystyle\int \frac{3(\arctan x)^2 - 1}{x^2+1}\,dx$;
   (b) $\displaystyle\int \frac{\sin x + 2\cos x}{2\sin x + \cos x}\,dx$.
7. **Matrices**. Find the inverse of the matrix
$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Selected answers and hints.

1. We have $\frac{dy}{dx} = (1-x^2)/(1+x^2)^2$, so $\frac{dy}{dx} = 0 \iff x = \pm 1$. Thus the stationary points are at $(1, 0.5)$ (a maximum) and $(-1, -0.5)$ (a minimum). The graph passes through the origin and tends to zero at $\pm\infty$.
2. $\frac{\partial f}{\partial x} = y^2 \cosh(x^2 y) + 2x^2 y^3 \sinh(x^2 y)$ and $\frac{\partial f}{\partial y} = 2xy \cosh(x^2 y) + x^3 y^2 \sinh(x^2 y)$.
3. (a) Firstly, $\sqrt{9-2x} = (9-2x)^{\frac{1}{2}} = 9^{\frac{1}{2}}(1 - \frac{2x}{9})^{\frac{1}{2}} = 3(1 + \frac{1}{2}(-\frac{2x}{9}) + \ldots)$, where all further terms have a factor of x^2. Thus
$$\lim_{x\to 0} \frac{\sqrt{9-2x} - 3}{x} = \lim_{x\to 0} \frac{(3 - \frac{x}{3} + \ldots) - 3}{x} = \lim_{x\to 0}\left(-\frac{1}{3} + \ldots\right) = -\frac{1}{3}.$$
   (b) The limit is of the form $\frac{0}{0}$, hence
$$\begin{aligned} \lim_{x\to 0} \frac{x - \tan x}{x - \sin x} &= \lim_{x\to 0} \frac{1 - \sec^2 x}{1 - \cos x} \quad \text{(by l'Hôpital's Rule)} \\ &= \lim_{x\to 0} \frac{\cos^2 x - 1}{\cos^2 x - \cos^3 x} \quad \text{(rearranging)} \\ &= \lim_{x\to 0} \frac{-2\cos x \sin x}{-2\cos x \sin x + 3\cos^2 x \sin x} \quad \text{(l'Hôpital again)} \\ &= \lim_{x\to 0} \frac{-2\cos x}{-2\cos x + 3\cos^2 x} \quad \text{(cancelling)} \\ &= -2. \end{aligned}$$
4. Writing $z_1 = a+ib$, $z_2 = c+id$ we have $\mathrm{Re}(z_1 z_2) = ac - bd$ and $\mathrm{Re}(z_1)\,\mathrm{Re}(z_2) = ac$. Thus $bd = 0$, so at least one of z_1 or z_2 must have zero imaginary part (i.e. is real).
5. The velocity vector is
$$\begin{aligned} \dot{\mathbf{r}} &= (12t\cos(t^2),\ -12t\sin(t^2),\ (3/2).(1+4t)^{1/2}.4) \\ &= (12t\cos(t^2),\ -12t\sin(t^2),\ 6(1+4t)^{1/2}). \end{aligned}$$
   Thus the speed is given by
$$\begin{aligned} |\dot{\mathbf{r}}| &= \sqrt{144t^2\cos^2(t^2) + 144t^2\sin^2(t^2) + 36(1+4t)} \\ &= 6\sqrt{(2t+1)^2} \\ &= 6(2t+1), \end{aligned}$$
   which varies linearly with t.
6. (a) Substituting $u = \tan^{-1} x$, $\int \frac{3(\tan^{-1} x)^2 - 1}{x^2+1} dx = (\tan^{-1} x)^3 - \tan^{-1} x + c$.

   (b) Using the standard substitution $t = \tan(x/2)$ and partial fractions, we find
$$\begin{aligned} \int \frac{\sin x + 2\cos x}{2\sin x + \cos x} dx &= \int \frac{-4t^2 + 4t + 4}{(1+t^2)(-t^2 + 4t + 1)} dt \\ &= \frac{1}{5}\int \left(\frac{-6t}{1+t^2} + \frac{8}{1+t^2} + \frac{-6t + 12}{-t^2 + 4t + 1} \right) dt \\ &= \frac{1}{5}\left(-3\ln(1+t^2) + 8\tan^{-1} t + 3\ln(|-t^2 + 4t + 1|)\right) + c. \end{aligned}$$
7. It turns out that $A^{-1} = A^T$.

For more details, start a thread on the discussion board.