

LAPLACE TRANSFORMS AND DIFFERENTIAL EQUATIONS

5 minute review. Recap the Laplace transform and the differentiation rule, and observe that this gives a good technique for solving linear differential equations: translating them to algebraic equations, and handling the initial conditions.

Class warm-up. Find a solution to the differential equation

$$\frac{dy}{dx} - 3y = e^{3x}$$

such that $y = 1$ when $x = 0$.

Problems. Choose from the below

1. **Inverse Laplace transforms.** Use the method of partial fractions where necessary to find the inverse Laplace transforms $f(t)$, $g(t)$ and $h(t)$ of the following:

$$F(s) = \frac{s+3}{s^2+6s+25}, \quad G(s) = \frac{6}{s^2-s-2}, \quad H(s) = \frac{2}{s^3+s^2+s+1}.$$

2. **A first-order example.** Solve the following differential equation using the Laplace transform:

$$\frac{dy}{dx} = xe^x + 2e^x + y, \quad \text{where } y = 3 \text{ when } x = 0.$$

3. **Some second-order examples.** Solve the following differential equation using the Laplace transform:

$$\begin{aligned} \frac{d^2y}{dx^2} + 9y &= 18e^{3x}, & \text{where } y = 0 \text{ and } \frac{dy}{dx} = 1 \text{ when } x = 0; \\ \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y &= 6xe^{2x}, & \text{where } y = 1 \text{ and } \frac{dy}{dx} = 2 \text{ when } x = 0. \end{aligned}$$

4. **A system of simultaneous differential equations*.**

Solve the following differential equations using the Laplace transform:

$$\frac{dx}{dt} = 4x + y, \quad \frac{dy}{dt} = 2x + 3y, \quad x(0) = 2, \quad y(0) = 5.$$

5. **Multiplying by t^* .** It can be shown that, if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(tf(t)) = -F'(s)$.

- Deduce from this that $\mathcal{L}(tf'(t)) = -sF'(s) - F(s)$ and $\mathcal{L}(tf''(t)) = f(0) - 2sF'(s) - s^2F''(s)$.
- Hence find a solution to the differential equation

$$x\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 3y = 0$$

such that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$.

For the warmup, the Laplace transform $Y(s)$ of $y(x)$ satisfies

$$sY(s) - y(0) - 3Y(s) = \frac{1}{s-3}.$$

Substituting in $y(0) = 1$ and rearranging, this means that

$$Y(s) = \frac{1 + \frac{1}{s-3}}{s-3} = \frac{1}{s-3} + \frac{1}{(s-3)^2}.$$

Using the shift rule, the inverse Laplace transform of this is $y(x) = e^{3x} + xe^{3x}$.

Selected answers and hints.

1.

$$\begin{aligned} F(s) &= \frac{s+3}{(s+3)^2+4^2}, & \text{so } f(t) &= e^{-3t} \cos(4t); \\ G(s) &= \frac{2}{s-2} - \frac{2}{s+1}, & \text{so } g(t) &= 2e^{2t} - 2e^{-t}; \\ H(s) &= \frac{1}{s+1} + \frac{1}{s^2+1} - \frac{s}{s^2+1}, & \text{so } h(t) &= e^{-t} + \sin(t) - \cos(t). \end{aligned}$$

2. The Laplace transform gives us $sY(s) - 3 = \frac{1}{(s-1)^2} + \frac{2}{s-1} + Y(s)$. Solving gives $Y(s) = \frac{3}{s-1} + \frac{2}{(s-1)^2} + \frac{1}{(s-1)^3}$, whence $y = e^x \left(3 + 2x + \frac{x^2}{2} \right)$.

3. For the first one, we get $s^2Y(s) - 1 + 9Y(s) = \frac{18}{s-3}$, and so $Y(s) = \frac{18}{(s-3)(s^2+9)} + \frac{1}{s^2+9} = \frac{1}{s-3} - \frac{s}{s^2+9} - \frac{2}{s^2+9}$, which means that $y = e^{3x} - \cos(3x) - \frac{2}{3} \sin(3x)$.

For the second one, we get $(s^2F(s) - s - 2) - 4(sF(s) - 1) + 4F(s) = 6/(s-2)^2$, which rearranges to give $F(s) = 1/(s-2) + 6/(s-2)^4$. So the answer is $y = (1 + x^3)e^{2x}$.

4. The Laplace transforms satisfy

$$sX(s) - 2 = 4X(s) + Y(s), \quad sY(s) - 5 = 2X(s) + 3Y(s).$$

Rearranging, we get

$$X(s) = \frac{Y(s) + 2}{s-4}, \quad Y(s) = \frac{2X(s) + 5}{s-3},$$

and then substituting in, we get

$$X(s) = \frac{\frac{2X(s)+5}{s-3} + 2}{s-4} = \frac{2X(s) + 5 + 2(s-3)}{(s-3)(s-4)},$$

and so

$$X(s) = \frac{\frac{2s-1}{(s-3)(s-4)}}{1 - \frac{2}{(s-3)(s-4)}} = \frac{2s-1}{s^2-7s+10} = \frac{2s-1}{(s-2)(s-5)} = \frac{3}{s-5} - \frac{1}{s-2},$$

and hence, by taking the inverse Laplace transform, $x(t) = 3e^{5t} - e^{2t}$. By a similar process, $y(t) = 3e^{5t} + 2e^{2t}$.

5. The Laplace transform gives $(3-s)F'(s) = 2F(s)$, which is separable with solution $F(s) = \frac{a}{(s-3)^2}$. Hence $y = axe^{3x}$, and using the initial conditions we find $a = 1$.

For more details, start a thread on the discussion board.