LAPLACE TRANSFORMS

5 minute review. Remind students of the definition of the Laplace transform $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)dt$, and go over the rules:

- the linearity rules: $\mathcal{L}(af(t)) = a\mathcal{L}(f(t))$, for a a constant, and $\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$,
- the shift rule: if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$, and
- the differentiation rule: if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(f'(t)) = sF(s) - f(0)$.

It might be good to mention the convention of using uppercase letters for Laplace transforms (F for f, G for g, and so on).

It might also possibly help to mention that a table of standard Laplace transforms will be in the exam formula booklet; it is reproduced at the end of the sheet.

Class warm-up. Compute the Laplace transform of $f(t) = 1$ by hand, and hence go over the Laplace transform of $f(t) = t$ (which was in the video).

Problems. Choose from the below.

1. **Using the rules**. Find, using the results of the formula booklet, the Laplace transforms of:

   $f(t) = 4\cos(2t);\qquad g(t) = t^5 e^{-t};\qquad h(t) = 5e^{4t}\sin(3t) + 2\cosh(7t).$

2. **The shift rule**. Check the shift rule for yourselves: in other words, show that, if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$.

3. **Transforms of** sin **and** cos.

   (a) Integrate by parts twice (integrating the trigonometric function and differentiating the exponential) to show that
   $$\int_0^\infty \sin(t)e^{-st}dt = 1 - s^2 \int_0^\infty \sin(t)e^{-st}dt.$$

   (b) Deduce that $\mathcal{L}(\sin(t)) = \frac{1}{1+s^2}$.

   (c) Use the differentiation rule to get a formula for $\mathcal{L}(\cos)$.

4. **Transforms of polynomials**. Continue the warm-up exercise to show that if $f(t) = t^n$, for $n = 2, 3, 4$, then its Laplace transform $F(s) = \mathcal{L}(f(t))$ is given by $F(s) = \frac{n!}{s^{n+1}}$. After these you should believe the general case! *(This can be done directly, by integration by parts, or indirectly, using the differentiation rule.)*

5. **Hyperbolic functions**.

   (a) Find formulae for the Laplace transforms of sinh and cosh, by following the strategy of Problem 2.

   (b) Find the same formulae directly from the definitions of sinh and cosh, using the linearity rules and the shift rule.

For the warm-up, when $f(t) = 1$ we have

$$F(s) = \int_0^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = 0 - \frac{1}{-s} = \frac{1}{s},$$

and then when $f(t) = t$ we have (by integrating by parts)

$$F(s) = \int_0^\infty te^{-st} dt = \left[t\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} dt = 0 + \frac{1}{s}\int_0^\infty e^{-st} dt,$$

and using the previous result, that's $\frac{1}{s^2}$.

Selected answers and hints.

1. They are:

$$F(s) = \frac{4s}{s^2+4}; \quad G(s) = \frac{120}{(s+1)^6}; \quad H(s) = \frac{15}{(s-4)^2+9} + \frac{2s}{s^2-49}.$$

2. We have

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{at}f(t)e^{-st} dt = \int_0^\infty f(t)e^{-(s-a)t} dt = F(s-a).$$

3. The differentiation rule gives us that $\mathcal{L}(\cos(t)) = \frac{s}{1+s^2}$.
4.
5. Differentiating twice and rearranging, as in Problem 2, gives that

$$\mathcal{L}(\sinh(t)) = \frac{1}{s^2-1},$$

and then the differentiation rule gives

$$\mathcal{L}(\cosh(t)) = \frac{s}{s^2-1}.$$

Alternatively,

$$\begin{aligned}\mathcal{L}(\sinh(t)) =& \mathcal{L}\left(\frac{e^t - e^{-t}}{2}\right) \quad \text{(by definition of sinh)} \\ =& \tfrac{1}{2}\left(\mathcal{L}(e^t) - \mathcal{L}(e^{-t})\right) \quad \text{(by linearity)} \\ =& \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) \quad \text{(by shift)} \\ =& \frac{1}{s^2-1},\end{aligned}$$

and a very similar calculation gives the corresponding result for cosh.

For more details, start a thread on the discussion board.

Table of Laplace transforms

Function $f(t)$	Laplace transform $F(s)$
t^n	$\frac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \ldots$)
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
$e^{at} f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$