

LAPLACE TRANSFORMS

5 minute review. Remind students of the definition of the Laplace transform $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t) dt$, and go over the rules:

- the linearity rules: $\mathcal{L}(af(t)) = a\mathcal{L}(f(t))$, for a a constant, and $\mathcal{L}(f(t)+g(t)) = \mathcal{L}(f(t)) + \mathcal{L}(g(t))$,
- the shift rule: if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$, and
- the differentiation rule: if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(f'(t)) = sF(s) - f(0)$.

It might be good to mention the convention of using uppercase letters for Laplace transforms (F for f , G for g , and so on).

It might also possibly help to mention that a table of standard Laplace transforms will be in the exam formula booklet; it is reproduced at the end of the sheet.

Class warm-up. Compute the Laplace transform of $f(t) = 1$ by hand, and hence go over the Laplace transform of $f(t) = t$ (which was in the video).

Problems. Choose from the below.

1. **Using the rules.** Find, using the results of the formula booklet, the Laplace transforms of:

$$f(t) = 4 \cos(2t); \quad g(t) = t^5 e^{-t}; \quad h(t) = 5e^{4t} \sin(3t) + 2 \cosh(7t).$$

2. **The shift rule.** Check the shift rule for yourselves: in other words, show that, if $\mathcal{L}(f(t)) = F(s)$, then $\mathcal{L}(e^{at}f(t)) = F(s-a)$.

3. **Transforms of sin and cos.**

- (a) Integrate by parts twice (integrating the trigonometric function and differentiating the exponential) to show that

$$\int_0^\infty \sin(t)e^{-st} dt = 1 - s^2 \int_0^\infty \sin(t)e^{-st} dt.$$

- (b) Deduce that $\mathcal{L}(\sin(t)) = \frac{1}{1+s^2}$.

- (c) Use the differentiation rule to get a formula for $\mathcal{L}(\cos)$.

4. **Transforms of polynomials.** Continue the warm-up exercise to show that if $f(t) = t^n$, for $n = 2, 3, 4$, then its Laplace transform $F(s) = \mathcal{L}(f(t))$ is given by $F(s) = \frac{n!}{s^{n+1}}$. After these you should believe the general case! (*This can be done directly, by integration by parts, or indirectly, using the differentiation rule.*)

5. **Hyperbolic functions.**

- (a) Find formulae for the Laplace transforms of \sinh and \cosh , by following the strategy of Problem 2.

- (b) Find the same formulae directly from the definitions of \sinh and \cosh , using the linearity rules and the shift rule.

For the warm-up, when $f(t) = 1$ we have

$$F(s) = \int_0^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} = 0 - \frac{1}{-s} = \frac{1}{s},$$

and then when $f(t) = t$ we have (by integrating by parts)

$$F(s) = \int_0^{\infty} te^{-st} dt = \left[t \frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} dt = 0 + \frac{1}{s} \int_0^{\infty} e^{-st} dt,$$

and using the previous result, that's $\frac{1}{s^2}$.

Selected answers and hints.

1. They are:

$$F(s) = \frac{4s}{s^2 + 4}; \quad G(s) = \frac{120}{(s+1)^6}; \quad H(s) = \frac{15}{(s-4)^2 + 9} + \frac{2s}{s^2 - 49}.$$

2. We have

$$\mathcal{L}(e^{at}f(t)) = \int_0^{\infty} e^{at}f(t)e^{-st} dt = \int_0^{\infty} f(t)e^{-(s-a)t} dt = F(s-a).$$

3. The differentiation rule gives us that $\mathcal{L}(\cos(t)) = \frac{s}{1+s^2}$.

4.

5. Differentiating twice and rearranging, as in Problem 2, gives that

$$\mathcal{L}(\sinh(t)) = \frac{1}{s^2 - 1},$$

and then the differentiation rule gives

$$\mathcal{L}(\cosh(t)) = \frac{s}{s^2 - 1}.$$

Alternatively,

$$\begin{aligned} \mathcal{L}(\sinh(t)) &= \mathcal{L}\left(\frac{e^t - e^{-t}}{2}\right) \quad (\text{by definition of sinh}) \\ &= \frac{1}{2} (\mathcal{L}(e^t) - \mathcal{L}(e^{-t})) \quad (\text{by linearity}) \\ &= \frac{1}{2} \left(\frac{1}{s-1} - \frac{1}{s+1} \right) \quad (\text{by shift}) \\ &= \frac{1}{s^2 - 1}, \end{aligned}$$

and a very similar calculation gives the corresponding result for cosh.

For more details, start a thread on the discussion board.

Table of Laplace transforms

<u>Function $f(t)$</u>	<u>Laplace transform $F(s)$</u>
t^n	$\frac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \dots$)
e^{at}	$\frac{1}{s - a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
$e^{at} f(t)$	$F(s - a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$