INHOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS

5 minute review. For an inhomogeneous second-order linear differential equation with constant coefficients

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x), \tag{1}$$

remind students that

- the *complementary function* is the general solution y_c to the homogeneous equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$;
- the *particular integral* is any solution y_p to equation (1), and this is found with sensible guesswork;
- the general solution to equation (1) is $y_c + y_p$.

Class warm-up. Call for some sensible suggestions for choices for particular integrals for the the cases below.

$$f(x) = x^2 - 3; \quad f(x) = e^{-5x}; \quad f(x) = \cos x; \quad f(x) = \sin(3x);$$
$$f(x) = x - e^x; \quad f(x) = x^{100} + \cos(100x) - e^{100x}.$$

Problems. Choose from the below.

1. **Standard examples.** Find the general solutions to the following differential equations.

   (a) $\frac{d^2y}{dx^2} - y = \sin x$;

   (b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 10x^3$;

   (c) $\frac{d^3y}{dx^3} - y = e^x$.

2. **A fourth-order equation?** Find the general solution to

$$2\frac{d^4y}{dx^4} + 2\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} = x.$$

   Hint: Start by integrating both sides of the equation twice (and remember your constants of integration!).

3. **A tenth-order equation?** Find the general solution to $\frac{d^2x}{dt^2} + x = 0$. Hence find a particular solution to

$$\frac{d^{10}x}{dt^{10}} + x = 2050e^{2t}$$

   for which $x = 3$ when $t = 0$ and $x = 1$ when $t = \frac{\pi}{2}$.

   NB: You are not asked to find the general solution of the tenth-order equation, just any particular solution which works.

For the warm-up, sensible choices for y_p are

- $ax^2 + bx + c$;
- Ae^{-5x};
- $a\cos x + b\sin x$;
- $a\sin(3x) + b\cos(3x)$;
- $ax + b + ce^x$;
- $a_{100}x^{100} + a_{99}x^{99} + \ldots + a_1x + a_0 + b\cos(100x) + c\sin(100x) + de^{100x}$.

Some of these suggestions could need modifying by adding in additional powers of x if it turns out that they form part of the complementary function. For example, the suggestion Ae^{-5x} will need to become Axe^{-5x} if -5 is a root of the auxiliary equation, or Ax^2e^{-5x} if -5 is a repeated root.

Selected answers and hints.

1. (a) The auxiliary equation is $k^2 - 1 = 0$. For the particular integral, try $y_p = a\sin x + b\cos x$. You should get something equivalent to $y = ae^x + be^{-x} - \frac{1}{2}\sin x$.

   (b) The auxiliary equation is $k^2 - 2k + 5 = 0$ and hence the complementary function is $y_c = e^x(a\cos(2x) + b\sin(2x))$. Try $y_p = ax^3 + bx^2 + cx + d$ for the particular integral and hence $y_p = 2x^3 + \frac{12}{5}x^2 - \frac{12}{25}x - \frac{144}{125}$.

   (c) The auxiliary equation is $k^3 - 1 = 0$ and hence the complementary function is $y_c = ae^x + e^{-x/2}(b\cos(\sqrt{3}x/2) + c\sin(\sqrt{3}x/2))$. Since e^x is in the complementary function we need a particular integral of the form axe^x and hence $y_p = \frac{1}{3}xe^x$.

2. By integrating twice, the equation becomes $2\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{1}{6}x^3 + c_1x + c_2$, which is now a standard second-order case. The auxiliary equation is $2k^2 + 2k + 1 = 0$ which has complex solutions. Choose a third order polynomial for the particular integral. The general solution turns out to be
$$y = e^{-\frac{1}{2}x}(A\cos(x/2) + B\sin(x/2)) + \frac{1}{6}x^3 - x^2 + Cx + D.$$

3. The general solution to the second-order equation is $x = A\cos t + B\sin t$. Now, if x is any solution to $\frac{d^2x}{dt^2} + x = 0$, then

   - $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} = 0$, so $\frac{d^4x}{dt^4} - x = 0$;
   - $\frac{d^6x}{dt^6} + \frac{d^4x}{dt^4} = 0$, so $\frac{d^6x}{dt^6} + x = 0$;
   - $\frac{d^8x}{dt^8} + \frac{d^6x}{dt^6} = 0$, so $\frac{d^8x}{dt^8} - x = 0$;
   - $\frac{d^{10}x}{dt^{10}} + \frac{d^8x}{dt^8} = 0$, so $\frac{d^{10}x}{dt^{10}} + x = 0$.

   Hence any solution to $\frac{d^2x}{dt^2} + x = 0$ is also a solution to the homogeneous equation $\frac{d^{10}x}{dt^{10}} + x = 0$. To solve $\frac{d^{10}x}{dt^{10}} + x = 2050e^{2t}$ we need a particular integral, and it turns out that $x = 2e^{2t}$ works. Thus $x = A\cos t + B\sin t + 2e^{2t}$ is a solution to the tenth-order equation. Putting in the initial conditions allows you to find A and B and leads to the particular solution $x = (1 - 2e^{\pi})\sin t + \cos t + 2e^{2t}$, which satisfies the requirements.

For more details, start a thread on the discussion board.