

HOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS

5 minute review. For the homogeneous second-order equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, cover how to solve it by looking at the roots of the *auxiliary equation* $ak^2 + bk + c = 0$. Deal with the cases of

- distinct real roots (solution: $y = Ae^{\alpha_1x} + Be^{\alpha_2x}$);
- complex roots (solution: $y = e^{\alpha x}(A \cos(\beta x) + B \sin(\beta x))$, where the roots are $\alpha \pm \beta i$; mention Euler's identity: $e^{i\theta} = \cos \theta + i \sin \theta$);
- repeated roots (solution: $y = Ae^{\alpha x} + Bxe^{\alpha x}$).

Class warm-up. Do a *first-order* equation by this method, e.g. $\frac{dy}{dx} - 7y = 0$, and observe separation of variables and integrating factors could both be used instead to reach the same result.

Move on to a standard second-order example, such as $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$.

Problems. Choose from the below.

1. **The three cases.** Solve each of the following homogeneous second-order linear differential equations:

(a) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 24y = 0,$

(b) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0,$

(c) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 26y = 0.$

2. **Derivatives of high order.** Find the general solution to the differential equation $\frac{d^2y}{dx^2} + 4y = 0$ and check that all these solutions are also solutions to the differential equation $\frac{d^6y}{dx^6} + 64y = 0$.

3. **Repeated roots.**

- (a) Can you find a 3-parameter family of solutions to

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0?$$

- (b) Can you construct a fourth-order differential equation with general solution $y = Ae^x + Bxe^x + Cx^2e^x + Dx^3e^x$?

- (c) What rule are you discovering here?

For the warm-up, the solutions are for the first part, aux eq: $k = 7$ hence $y = ae^{7x}$, or by separation of variables $\int \frac{1}{y} dy = \int 7 dx$ or integrating factor e^{-7x} ; and for the second part, aux eq: $k^2 + k + 1 = 0$ hence $y = e^{-\frac{1}{2}x}(a \cos(\frac{\sqrt{3}}{2}x) + b \sin(\frac{\sqrt{3}}{2}x))$.

Selected answers and hints.

1. You should get something equivalent to this:

(a) Auxiliary equation $k^2 - 10k + 24 = 0$ hence $y = ae^{6x} + be^{4x}$,

(b) Auxiliary equation $k^2 - 10k + 25 = 0$ hence $y = (a + bx)e^{5x}$,

(c) Auxiliary equation $k^2 - 10k + 26 = 0$ hence $y = e^{5x}(a \cos x + b \sin x)$.

2. The auxiliary equation is $k^2 + 4 = 0$, hence the general solution is $y = a \cos(2x) + b \sin(2x)$.

We could differentiate six times to check this, but it's faster to notice that if we have a solution to $\frac{d^2y}{dx^2} + 4y = 0$, then $\frac{d^2y}{dx^2} = -4y$ and, differentiating repeatedly,

$$\begin{aligned}\frac{d^3y}{dx^3} &= -4 \frac{dy}{dx}; \\ \frac{d^4y}{dx^4} &= -4 \frac{d^2y}{dx^2} = -4(-4y) = 16y; \\ \frac{d^5y}{dx^5} &= 16 \frac{dy}{dx}; \\ \frac{d^6y}{dx^6} &= 16 \frac{d^2y}{dx^2} = 16(-4y) = -64y,\end{aligned}$$

so $\frac{d^6y}{dx^6} + 64y = 0$.

3. (a) As the auxiliary equation is $k^3 + 3k^2 + 3k + 1 = 0$, this is $(k + 1)^3 = 0$, so there is repeated root $k = -1$ with multiplicity 3. It turns out that $y = e^{-x}$, $y = xe^{-x}$ and $y = x^2e^{-x}$ are all solutions, so the general solution is $y = ae^{-x} + bxe^{-x} + cx^2e^{-x}$.

(b) Following the pattern in (a) we might expect a repeated root $k = +1$ with multiplicity 4. So $(m-1)^4$ is the auxiliary equation. The given expression will therefore be the general solution to $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$.

(c) If $k = \alpha$ is a repeated root with multiplicity m of the auxiliary equation for a linear, homogenous differential equation with constant coefficients, then $y = e^{\alpha x}$, $y = xe^{\alpha x}$, \dots , $y = x^{m-1}e^{\alpha x}$ are all solutions.

For more details, start a thread on the discussion board.