HOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS

5 minute review. For the homogeneous second-order equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$, cover how to solve it by looking at the roots of the *auxiliary equation* $ak^2+bk+c=0$. Deal with the cases of

- distinct real roots (solution: $y=Ae^{\alpha_1 x}+Be^{\alpha_2 x}$);
- complex roots (solution: $y=e^{\alpha x}(A\cos(\beta x)+B\sin(\beta x))$, where the roots are $\alpha\pm\beta i$; mention Euler's identity: $e^{i\theta}=\cos\theta+i\sin\theta$);
- repeated roots (solution: $y=Ae^{\alpha x}+Bxe^{\alpha x}$).

Class warm-up. Do a *first-order* equation by this method, e.g. $\frac{dy}{dx}-7y=0$, and observe separation of variables and integrating factors could both be used instead to reach the same result.

Move on to a standard second-order example, such as $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$.

Problems. Choose from the below.

1. **The three cases**. Solve each of the following homogeneous second-order linear differential equations:

   (a) $$\frac{d^2y}{dx^2}-10\frac{dy}{dx}+24y=0,$$

   (b) $$\frac{d^2y}{dx^2}-10\frac{dy}{dx}+25y=0,$$

   (c) $$\frac{d^2y}{dx^2}-10\frac{dy}{dx}+26y=0.$$

2. **Derivatives of high order**. Find the general solution to the differential equation $\frac{d^2y}{dx^2}+4y=0$ and check that all these solutions are also solutions to the differential equation $\frac{d^6y}{dx^6}+64y=0$.

3. **Repeated roots**.

   (a) Can you find a 3-parameter family of solutions to
   $$\frac{d^3y}{dx^3}+3\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=0?$$

   (b) Can you construct a fourth-order differential equation with general solution $y=Ae^x+Bxe^x+Cx^2e^x+Dx^3e^x$?

   (c) What rule are you discovering here?

For the warm-up, the solutions are for the first part, aux eq: $k = 7$ hence $y = ae^{7x}$, or by separation of variables $\int \frac{1}{y}\,dy = \int 7\,dx$ or integrating factor e^{-7x}; and for the second part, aux eq: $k^2 + k + 1 = 0$ hence $y = e^{-\frac{1}{2}x}(a\cos(\frac{\sqrt{3}}{2}x) + b\sin(\frac{\sqrt{3}}{2}x))$.

Selected answers and hints.

1. You should get something equivalent to this:
   (a) Auxiliary equation $k^2 - 10k + 24 = 0$ hence $y = ae^{6x} + be^{4x}$,
   (b) Auxiliary equation $k^2 - 10k + 25 = 0$ hence $y = (a + bx)e^{5x}$,
   (c) Auxiliary equation $k^2 - 10k + 26 = 0$ hence $y = e^{5x}(a\cos x + b\sin x)$.
2. The auxiliary equation is $k^2 + 4 = 0$, hence the general solution is $y = a\cos(2x) + b\sin(2x)$.

   We could differentiate six times to check this, but it's faster to notice that if we have a solution to $\frac{d^2y}{dx^2} + 4y = 0$, then $\frac{d^2y}{dx^2} = -4y$ and, differentiating repeatedly,
$$\begin{aligned}
\frac{d^3y}{dx^3} &= -4\frac{dy}{dx};\\
\frac{d^4y}{dx^4} &= -4\frac{d^2y}{dx^2} = -4(-4y) = 16y;\\
\frac{d^5y}{dx^5} &= 16\frac{dy}{dx};\\
\frac{d^6y}{dx^6} &= 16\frac{d^2y}{dx^2} = 16(-4y) = -64y,
\end{aligned}$$
   so $\frac{d^6y}{dx^6} + 64y = 0$.
3. (a) As the auxiliary equation is $k^3 + 3k^2 + 3k + 1 = 0$, this is $(k + 1)^3 = 0$, so there is repeated root $k = -1$ with multiplicity 3. It turns out that $y = e^{-x}$, $y = xe^{-x}$ and $y = x^2e^{-x}$ are all solutions, so the general solution is $y = ae^{-x} + bxe^{-x} + cx^2e^{-x}$.
   (b) Following the pattern in (a) we might expect a repeated root $k = +1$ with multiplicity 4. So $(m-1)^4$ is the auxiliary equation. The given expression will therefore be the general solution to $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$.
   (c) If $k = \alpha$ is a repeated root with multiplicity m of the auxiliary equation for a linear, homogenous differential equation with constant coefficients, then $y = e^{\alpha x}$, $y = xe^{\alpha x}$, ..., $y = x^{m-1}e^{\alpha x}$ are all solutions.

For more details, start a thread on the discussion board.