INTEGRATING FACTORS

5 minute review. Remind students how to use integrating factors to solve linear first-order differential equations, namely that if $\frac{dy}{dx} + P(x)y = Q(x)$, then the integrating factor is $I(x) = \exp(\int P(x)dx)$, and multiplying through by $I(x)$ makes the left-hand side into an exact derivative. The example $\frac{dy}{dx} + \frac{y}{x} = x^3$ is a good one, if needed.

Class warm-up. Find the solution to

$$\frac{dy}{dx} + 8x^7 y = e^{x-x^8},$$

satisfying $y = 2$ when $x = 1$.

Problems. Choose from the below.

1. **A basic example**. Find the general solution to the differential equation
$$\frac{dy}{dx} + \frac{2y}{x} = \frac{\sin x}{x^2}.$$
2. **An example with initial conditions**. Find the solution of the differential equation
$$\exp\left(\frac{dy}{dx} + \frac{y}{x}\right) = x$$
which satisfies $x = 1$ at $y = 1$.
3. **Daily temperature cycles**. Recall Newton's law of cooling, which gives a differential equation describing the temperature of a body T in terms of the ambient temperature T_a, namely
$$\frac{dT}{dt} = -k(T - T_a).$$
In the videos we solved this differential equation when T_a was constant. Instead, let's assume $T_a = \sin t$ (a crude approximation of a daily temperature cycle). Find the solution of the resulting differential equation.
4. **A non-linear equation**. By considering the expansion of $(s-t)(s+t-u)$, find two independent families of solutions to
$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y^2 + xy = 0.$$

For the review, the integrating factor is x and the solution is $y = \frac{x^4}{5} + \frac{C}{x}$.

For the warm-up, the integrating factor is e^{x^8}, and the particular solution is $y = e^{x-x^8} + e^{1-x^8}$.

Selected answers and hints.

1. The integrating factor is x^2 and the solution is
$$y = \frac{c - \cos x}{x^2}.$$
2. The integrating factor is x: integration by parts is needed and the solution turns out to be
$$y = \frac{1}{2}x \ln x - \frac{1}{4}x + \frac{5}{4x}.$$
3. The integrating factor is (as in the example considered in the videos) e^{kt}, and integration by parts is needed to find $\int e^{kt} \sin t dt$. Don't keep integrating for ever! Instead combine terms on the LHS and RHS. The solutions turn out to be
$$T = \frac{k(k \sin t - \cos t)}{k^2 + 1} + ce^{-kt},$$
where c is a constant.
4. The equation factorises as $\left(\frac{dy}{dx} - y\right)\left(\frac{dy}{dx} + y - x\right) = 0$, and we can find one family of solutions for each bracket. The first is solved by separating variables, the second has an integrating factor of e^x. This leads to the solutions $y = Ae^x$ and $y = x - 1 + Be^{-x}$.

For more details, start a thread on the discussion board.