DIFFERENTIAL EQUATIONS: THE BASICS AND SEPARATION OF VARIABLES

5 minute review. Remind students what a differential equation is, the difference between ordinary and partial, linear and non-linear, and what the order of a differential equation is. (You may wish to mention applications; a few are described overleaf.) Also, remind students of the principle of separation of variables for solving equations of the form $\frac{dy}{dx} = f(x)g(y)$, using the warm-up below.

Class warm-up. 1. Differentiate the following functions of x, and construct a linear first-order ordinary differential equation containing a y-term whose solution is the given function. (a) $y = x^{100}$, (b) $y = e^{e^x}$. 2. Find the general solution to $\dfrac{dy}{dx} - 2x\sqrt{1-y^2} = 0$.

Problems. Choose from the below.

1. **Differential equations for** sin **and** cos.
   (a) Show that, for constants a and b, the function $y = a\cos x + b\sin x$ is a solution to the differential equation
   $$\frac{d^2y}{dx^2} + y = 0. \tag{1}$$
   (b) Find values for a and b in order to give a solution to Equation (1) satisfying $y = 3$ and $\frac{dy}{dx} = 4$ when $x = 0$.
   (c) For which a and b are those solutions to Equation (1) also solutions to the differential equation $\left(\frac{dy}{dx}\right)^2 + y^2 = 1$?
2. (a) Find a solution to
   $$\frac{dy}{dx} = e^{y-x}$$
   where $y = 1$ at $x = 1$.
   (b) Find a solution to
   $$x = \ln\left(\frac{dy}{dx}\right) - \ln(y),$$
   where $y = e$ at $x = 0$.
3. **Monomials**. Let m and n be integers. Find the general solution to the differential equation
   $$y^n \frac{dy}{dx} = x^m.$$
   There are four families of cases to pay attention to: (a) $m = n = -1$; (b) $m = -1$, $n \neq -1$; (c) $m \neq -1$, $n = -1$; (d) both m and n are different to -1.
4. **A substitution**. Find the general solution to the differential equation
   $$(yx^2 + x^3)\frac{dy}{dx} = y^3 + 2y^2x - x^3.$$
   (Hint: use the substitution $v = \frac{y}{x}$ and solve for v, and hence for y.)

Applications include Newton's second Law, force = mass $\times$ acceleration, which is often a 2nd-order differential equation, depending on nature of the force. Alternatively, there is "force is the rate of change of momentum" which often gives a 1st-order equation. The linear theory of elasticity for compressive load on a beam (with an equal force, F, applied at both ends) where y is displacement perpendicular to the beam and x is distance along the beam and we have $\frac{d^2y}{dx^2} \propto -Fy$.

For the warm-up, 1. (a) is a solution to $\frac{dy}{dx} = \frac{100}{x}y$ and (b) to $\frac{dy}{dx} = ye^x$. 2. the equations separate to give $\int \frac{1}{\sqrt{1-y^2}}\,dy = \int 2x\,dx$ and the general solution is $y = \sin(x^2+c)$, for any $c \in \mathbb{R}$. The solutions mostly involve substituting the expressions given for y into the differential equations.

Selected answers and hints.

1. For part (b) when $x = 0$, $y = a = 3$ and $y' = b = 4$. In part (c) you should find the condition that $a^2 + b^2 = 1$.
2. (a) Write $e^{y-x} = e^y/e^x$ and rearrange. The solution $y = x$ is what's aimed for!

   (b) Exponentiate both sides and rearrange to get $\frac{1}{y}\frac{dy}{dx} = e^x$; you should find the general solution $y = e^{c+e^x}$, or equivalently $y = ae^{e^x}$. Putting $x = 0$, $y = e$ gives $a = 1$.
3. Three of the 4 cases refer to different ways in which a zero could arise in the denominator when you integrate as is. The following are valid forms for the answers in each case:

   (a) $y = ax$;

   (b) $y^{n+1} = (n+1)\ln|x| + c$;

   (c) $y = ae^{x^{m+1}/(m+1)}$;

   (d) $y^{n+1} = \frac{n+1}{m+1}x^{m+1} + c$.
4. Dividing through by x^3, the equation becomes $(\frac{y}{x}+1)\frac{dy}{dx} = (\frac{y}{x})^3 + 2(\frac{y}{x})^2 - 1$. Letting $v = \frac{y}{x}$, it follows that $y = vx$ and hence $\frac{dy}{dx} = v + x\frac{dv}{dx}$. After making the substitution we find that either $v + 1 = 0$ (which gives the solution $y = -x$) or $\frac{dv}{dx} = (v+1)(v-1)/x$, which is separable with general solution $y = x(1+Ax^2)/(1-Ax^2)$ for any $A \in \mathbb{R}$. The v integral requires integration by parts.

For more details, start a thread on the discussion board.