MATRICES: EIGENVALUES AND EIGENVECTORS

5 minute review. Remind students how to find eigenvalues using the characteristic equation, and how to find the eigenvectors associated with a given eigenvalue.

Class warm-up. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}.$$

Find an eigenvector associated to each eigenvalue.

Problems. Choose from the below.

1. **Eigenvalues and eigenvectors**. Find the characteristic equation, eigenvectors, and eigenvalues for the following matrices.

$$A = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 5 & -1 \\ 1 & 7 \end{bmatrix}, C = \begin{bmatrix} 5 & 12 \\ 1 & 4 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}, F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

2. **Reversing the problem***. In each case, find a 2×2 matrix with the given information:
   (a) eigenvalues 3 and -4,
   (b) eigenvalue 2 for eigenvector $[1 \quad -1]^T$,
   (c) eigenvalue -3 for eigenvector $[2 \quad 1]^T$ and eigenvalue 6,
   (d) eigenvectors $[1 \quad 3]^T$ and $[6 \quad -2]^T$, and
   (e) eigenvector $[3 \quad 2]^T$ and eigenvalue 5 for eigenvector $[1 \quad 1]^T$.

For the warm-up, find the eigenvalues, λ, from the characteristic polynomial obtained from $|A-\lambda I|=0$. Then solve $(A-\lambda I)\mathbf{x}=0$ to get the eigenvectors, $\mathbf{x}$, for each λ.

For the example given, the characteristic polynomial is $\lambda^2-3\lambda-10$, leading to eigenvalues -2, 5 and corresponding eigenvectors $[1 \quad 1]^T$, $[-4/3 \quad 1]^T$.

Selected answers and hints.

1. (a) A: $\lambda^2-4\lambda-21=0$, eigenvalues $7,-3$ with eigenvectors $[1 \quad 1]^T$ and $[\frac{-3}{2} \quad 1]^T$
   (b) B: $\lambda^2-12\lambda+36=0$, eigenvalue 6 with eigenvector $[1 \quad -1]^T$
   (c) C: $\lambda^2-9\lambda+8=0$, eigenvalues $1,8$ with eigenvectors $[-3 \quad 1]^T$ and $[4 \quad 1]^T$
   (d) D: $\lambda^3+\lambda^2-12\lambda=0$, eigenvalues $0,3,-4$ with eigenvectors $[1 \quad 6 \quad -13]^T$, $[2 \quad 3 \quad -2]^T$ and $[1 \quad -2 \quad -1]^T$
   (e) E: $\lambda^3-7\lambda^2+36=0$, eigenvalues $-2,6,3$ with eigenvectors $[1 \quad 0 \quad -1]^T$, $[1 \quad 2 \quad 1]^T$ and $[1 \quad -1 \quad 1]^T$
   (f) F: $\lambda^3-3\lambda-2=0$, eigenvalues $2,-1$ with eigenvector $[1 \quad 1 \quad 1]^T$ associated to 2, and $[1 \quad 0 \quad -1]^T$, $[-1 \quad 1 \quad 0]^T$ associated to -1
2. Remember that eigenvalues, λ, and eigenvectors, $\begin{bmatrix} x & y \end{bmatrix}^T$ of an arbitrary 2x2 matrix are found from $\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

   You may have other solutions.

   (a)
   $$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$
   (b)
   $$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$
   (c)
   $$\begin{bmatrix} 0 & -6 \\ -3 & 3 \end{bmatrix}$$
   (d)
   $$\begin{bmatrix} \frac{19}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{11}{10} \end{bmatrix}, \quad \text{or even} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
   (e)
   $$\begin{bmatrix} 2 & 3 \\ -2 & 7 \end{bmatrix}$$

For more details, start a thread on the discussion board.