

MATRICES: EIGENVALUES AND EIGENVECTORS

5 minute review. Remind students how to find eigenvalues using the characteristic equation, and how to find the eigenvectors associated with a given eigenvalue.

Class warm-up. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & -4 \\ -3 & 1 \end{bmatrix}.$$

Find an eigenvector associated to each eigenvalue.

Problems. Choose from the below.

- 1. Eigenvalues and eigenvectors.** Find the characteristic equation, eigenvectors, and eigenvalues for the following matrices.

$$A = \begin{bmatrix} 1 & 6 \\ 4 & 3 \end{bmatrix}, B = \begin{bmatrix} 5 & -1 \\ 1 & 7 \end{bmatrix}, C = \begin{bmatrix} 5 & 12 \\ 1 & 4 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}, E = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}, F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- 2. Reversing the problem***. In each case, find a 2×2 matrix with the given information:

- (a) eigenvalues 3 and -4 ,
- (b) eigenvalue 2 for eigenvector $[1 \quad -1]^T$,
- (c) eigenvalue -3 for eigenvector $[2 \quad 1]^T$ and eigenvalue 6,
- (d) eigenvectors $[1 \quad 3]^T$ and $[6 \quad -2]^T$, and
- (e) eigenvector $[3 \quad 2]^T$ and eigenvalue 5 for eigenvector $[1 \quad 1]^T$.

For the warm-up, find the eigenvalues, λ , from the characteristic polynomial obtained from $|A - \lambda I| = 0$. Then solve $(A - \lambda I)\mathbf{x} = 0$ to get the eigenvectors, \mathbf{x} , for each λ .

For the example given, the characteristic polynomial is $\lambda^2 - 3\lambda - 10$, leading to eigenvalues $-2, 5$ and corresponding eigenvectors $[1 \ 1]^T, [-4/3 \ 1]^T$.

Selected answers and hints.

1. (a) A : $\lambda^2 - 4\lambda - 21 = 0$, eigenvalues $7, -3$ with eigenvectors $[1 \ 1]^T$ and $[\frac{-3}{2} \ 1]^T$
 - (b) B : $\lambda^2 - 12\lambda + 36 = 0$, eigenvalue 6 with eigenvector $[1 \ -1]^T$
 - (c) C : $\lambda^2 - 9\lambda + 8 = 0$, eigenvalues $1, 8$ with eigenvectors $[-3 \ 1]^T$ and $[4 \ 1]^T$
 - (d) D : $\lambda^3 + \lambda^2 - 12\lambda = 0$, eigenvalues $0, 3, -4$ with eigenvectors $[1 \ 6 \ -13]^T, [2 \ 3 \ -2]^T$ and $[1 \ -2 \ -1]^T$
 - (e) E : $\lambda^3 - 7\lambda^2 + 36 = 0$, eigenvalues $-2, 6, 3$ with eigenvectors $[1 \ 0 \ -1]^T, [1 \ 2 \ 1]^T$ and $[1 \ -1 \ 1]^T$
 - (f) F : $\lambda^3 - 3\lambda - 2 = 0$, eigenvalues $2, -1$ with eigenvector $[1 \ 1 \ 1]^T$ associated to 2 , and $[1 \ 0 \ -1]^T, [-1 \ 1 \ 0]^T$ associated to -1
2. Remember that eigenvalues, λ , and eigenvectors, $[x \ y]^T$ of an arbitrary 2×2 matrix are found from $\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

You may have other solutions.

(a)

$$\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 & -6 \\ -3 & 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} \frac{19}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{19}{10} \end{bmatrix}, \quad \text{or even} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(e)

$$\begin{bmatrix} 2 & 3 \\ -2 & 7 \end{bmatrix}$$

For more details, start a thread on the discussion board.