MATRICES: SYSTEMS OF EQUATIONS

Announcement. Please remind students that there is a full-class lecture in either Week 7 (MAS140,151) or 8 (MAS152,156) on exam technique.

5 minute review. Remind students about the different types of systems of equations: homogeneous versus non-homogeneous, and singular versus non-singular. Recall how many solutions there are in each case: non-singular = exactly one (which is 0 if homogeneous), singular + non-homogeneous = zero (inconsistent) or infinitely many (use parameters to describe), singular + homogeneous = infinitely many (since 0 is a solution).

Class warm-up. Discuss what kinds of systems each of these are, and therefore the nature of their solution sets. Find the solutions.

$$\text{(a)}\ \begin{aligned} 2x - 5y &= 1 \\ x + 3y &= 6 \end{aligned} \qquad \text{(b)}\ \begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 - 4x_3 &= 0 \\ 5x_1 - x_2 - 2x_3 &= 0 \end{aligned}$$

Problems. Choose from the below.

1. **Standard systems of equations**. Solve the following systems of equations, using matrix methods where possible.

$$\text{(a)}\ \begin{aligned} 2x_1 + 7x_3 &= 0 \\ x_1 - x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 2x_3 &= 0 \end{aligned} \qquad \text{(b)}\ \begin{aligned} 2x_1 + 3x_3 &= 1 \\ x_1 - x_2 + 5x_3 &= 1 \\ 2x_2 - 4x_3 &= -2 \end{aligned}$$

$$\text{(c)}\ \begin{aligned} 2x_1 + 7x_3 &= 29 \\ x_1 - x_2 + 3x_3 &= 0 \\ 4x_1 + 5x_2 + 2x_3 &= -29 \end{aligned} \qquad \text{(d)}\ \begin{aligned} 2x_1 + 3x_3 &= 0 \\ x_1 - x_2 + 5x_3 &= 0 \\ 2x_2 - 4x_3 &= 0 \end{aligned}$$

$$\text{(e)}\ \begin{aligned} x_1 + 2x_2 + 6x_3 + 5x_4 &= 1 \\ x_2 - 2x_3 + 4x_4 &= 0 \\ x_3 + 3x_4 &= 0 \\ 3x_1 + x_4 &= 2 \end{aligned} \qquad \text{(f)}\ \begin{aligned} 8x_1 - x_2 + 3x_3 &= 7 \\ 5x_1 + 2x_2 + x_3 &= 0 \\ x_1 + x_2 + 7x_3 &= 0 \\ 2x_1 - 4x_2 - 5x_3 &= 7 \end{aligned}$$

2. **Singular systems**. Find the value(s) of α for which the following systems of equations have infinitely many solutions and then find those solutions, using matrix methods where possible.

$$\text{(a)}\ \begin{aligned} -x_1 + 4x_2 + \alpha x_3 &= -2 \\ 2x_1 - x_2 &= -1 \\ 5x_1 + x_2 - x_3 &= -5 \end{aligned} \qquad \text{(b)}\ \begin{aligned} 6x_1 + 3x_2 - 7x_3 &= 1 \\ x_1 + 4x_2 + x_3 &= -1 \\ 9x_1 + 15x_2 - 4x_3 &= \alpha \end{aligned}$$

$$\text{(c)}\ \begin{aligned} (\alpha + 1)x_1 + 6x_2 + 5x_3 &= 6 \\ (\alpha - 2)x_2 + x_3 &= -3 \\ (\alpha - 3)x_3 &= 0 \end{aligned}$$

3. **Linear combinations revisited***. Let A and B be any 2×2 matrices. Investigate whether you can find scalars $\alpha, \beta, \gamma, \delta$ such that

$$\alpha A + \beta A^T + \gamma B + \delta B^T = I_2.$$

For the warm-up, (a) is non-homogeneous and $|A| = 11$, so is non-singular. $A^{-1} = \frac{1}{11}\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$, therefore the solution is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

For (b), $|A| = 0$, and so it's singular. The equations reduce to:

$$x_1 - x_2 + x_3 = 0, \quad 4x_1 - 3x_3 = 0$$

Hence the solution is $\mathbf{x} = [\frac{3}{4}\lambda \quad \frac{7}{4}\lambda \quad \lambda]$.

Selected answers and hints.

1. (a) $|A| = 29 \neq 0$ so non-singular and hence the only solution is $[0 \quad 0 \quad 0]^T$.

   (b) $|A| = -6$ hence non-singular so an inverse can be obtained. $A^{-1} = -\frac{1}{6}\begin{bmatrix} -6 & 6 & 3 \\ 4 & -8 & -7 \\ 2 & -4 & -2 \end{bmatrix}$. Hence the solution is $[1 \quad \frac{-5}{3} \quad \frac{-1}{3}]^T$.

   (c) $|A| = 29$, and $A^{-1} = \frac{1}{29}\begin{bmatrix} -17 & 35 & 7 \\ 10 & -24 & 1 \\ 9 & -10 & -2 \end{bmatrix}$; the solution is $[-24 \quad 9 \quad 11]^T$.

   (d) $|A| = -6$ and is thus non-singular; the only solution is $[0 \quad 0 \quad 0]^T$.

   (e) $|A| = 100$. It is probably easiest to solve this one directly rather than finding the inverse to get $[\frac{67}{100} \quad \frac{1}{10} \quad \frac{3}{100} \quad \frac{-1}{100}]^T$.

   (f) Note that row 4 = row 1 − row 2 − row 3, hence we can solve the 3x3 system including only the first three equations. $|A| = 147$. Either find the inverse $A^{-1} = \frac{1}{147}\begin{bmatrix} 13 & 4 & -7 \\ -34 & 53 & 7 \\ 3 & -9 & 21 \end{bmatrix}$ or solve directly to find $[\frac{13}{21} \quad \frac{-34}{21} \quad \frac{1}{7}]^T$.

2. In cases (a) and (c), we can find α using $|A| = 0$; the fact that $|A| = 0$ also means that A has no inverse and so we solve the equations directly.

   (a) $\alpha = -1$; the solutions are $x_1 = \lambda, x_2 = 2\lambda + 1, x_3 = 7\lambda + 6$.

   (b) $\alpha = -2$ is necessary to make the equations consistent (e.g. Eq 1 + 3*Eq 2 = Eq 3); the solutions are $x_1 = \lambda, x_2 = \frac{-13\lambda - 6}{31}, x_3 = \frac{21\lambda - 7}{31}$

   (c) The possible values of α are $-1, 2, 3$. There are no solutions when $\alpha = 2$ because the equations are inconsistent.

   - if $\alpha = -1$, the solutions are $[\lambda \quad 1 \quad 0]^T$;
   - if $\alpha = 3$, the solutions are $[\lambda \quad 21 - 4\lambda \quad -24 + 4\lambda]^T$.

3. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. Writing out the four equations that result gives

$$\begin{bmatrix} a & a & e & e \\ b & c & f & g \\ c & b & g & f \\ d & d & h & h \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

   which is a singular system. This means there is either no solution or infinitely many solutions, depending on the matrices A and B.

For more details, start a thread on the discussion board.