

## MATRICES: MORE INVERSES

**5 minute review.** Remind students how to compute the cofactors  $A_{ij} = (-1)^{i+j} \det(M_{ij})$ , where  $M_{ij}$  is the matrix obtained from  $A$  by deleting row  $i$  and column  $j$ . The adjoint of  $A$  is then defined to be the transpose of the matrix of cofactors,  $\text{adj}(A) = [A_{ij}]^T$ , and the inverse of  $A$  (assuming  $\det(A) \neq 0$ ) is given by  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$ .

**Class warm-up.** Find the matrix of cofactors, the determinant, and the inverse for the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 9 & -1 & 2 \\ 0 & 3 & 3 \end{bmatrix}.$$

**Problems.** Choose from the below.

1. **Finding more inverses.** For each of the matrices below, find the matrix of cofactors, the determinant, and the inverse.

$$A = \begin{bmatrix} 1 & 0 & 6 \\ -2 & 2 & 1 \\ -3 & 5 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 3 & -5 \\ 11 & 9 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 1 & 1 \\ -1 & 4 & 4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

2. **Solving matrix equations.** Solve for the matrix  $B_i$  in each equation below. Note the size and shape!

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & -1 \\ -3 & 4 & -2 \end{bmatrix} B_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} B_2 = \begin{bmatrix} 2 & 4 & 5 \\ 1 & -1 & 0 \\ 6 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} B_3 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 6 & 1 & 8 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 8 & 0 \\ -3 & 2 & 0 & 4 \\ -2 & -4 & -8 & 0 \end{bmatrix}$$

3. **Transpose and inverse.** Here we investigate  $(A^T)^{-1}$  and  $(A^{-1})^T$ .

- (a) Start with a  $2 \times 2$  matrix of your choice, find its inverse, and then take the transpose; now start with the same matrix, take the transpose, and find the inverse of that matrix. Now try it with a simple  $3 \times 3$  matrix.
- (b) Find the matrix of cofactors for  $A^T$ . How does it relate to the matrix of cofactors for  $A$ ?
- (c) Use the formula for the inverse with cofactors and determinants to show that  $(A^T)^{-1} = (A^{-1})^T$ .

- (d) Recall that a symmetric matrix  $A$  is one that satisfies  $A = A^T$ . An example is

$$A = \begin{bmatrix} 1 & 0 & 6 \\ 0 & 3 & 2 \\ 6 & 2 & -2 \end{bmatrix}.$$

Find the inverse of  $A$ , and check that  $A^{-1}$  is also symmetric. Can you show that if a symmetric matrix  $A$  is invertible, then  $A^{-1}$  is also symmetric using the cofactor formula for the inverse?

- (e) A skew symmetric matrix  $B$  is one that satisfies  $-B = B^T$ . An example is

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Find the inverse of  $B$ , and check that  $B^{-1}$  is also skew symmetric. Now try a  $3 \times 3$  skew symmetric matrix and see what happens.

For the warm-up,  $A^{-1} = \frac{1}{-63} \begin{bmatrix} -9 & -6 & 16 \\ -27 & 3 & 34 \\ 27 & -3 & -55 \end{bmatrix}$ .

**Selected answers and hints.**

1.

$$A^{-1} = \frac{1}{-29} \begin{bmatrix} -5 & 30 & -12 \\ -3 & 18 & -13 \\ -4 & -5 & 2 \end{bmatrix}$$

$B^{-1}$  does not exist since  $|B| = 0$ .

$$C^{-1} = \frac{1}{-22} \begin{bmatrix} -16 & -8 & 6 & 16 \\ 2 & -10 & 2 & -2 \\ -3 & 4 & -3 & -8 \\ -3 & 4 & -3 & 14 \end{bmatrix}$$

2. For any equation  $AB = C$ , to solve for  $B$  we multiply by  $A^{-1}$  to get  $B = A^{-1}C$ .

$$B_1 = \begin{bmatrix} -12 \\ 17 \\ 51 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & -114 & -120 \\ 1 & 95 & 100 \\ 0 & -24 & -25 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \frac{51}{2} & -8 & 4 & 0 \\ -49 & 12 & -4 & 2 \\ -13 & 4 & -2 & 0 \end{bmatrix}$$

3. (a) We will see later that  $(A^T)^{-1} = (A^{-1})^T$ .

(b) The cofactor  $A_{ij}$  is the same as the  $(j, i)$  cofactor of  $A^T$ :  $A_{ij} = (A^T)_{ji}$ .

(c) This follows from the previous part plus the fact that  $|A| = |A^T|$ .

(d) For the matrix given,  $A^{-1} = \frac{1}{-118} \begin{bmatrix} -10 & 12 & -18 \\ 12 & -38 & -2 \\ -18 & -2 & 3 \end{bmatrix}$ .

In general, if  $A$  is invertible, then  $(A^{-1})^T = (A^T)^{-1} = A^{-1}$  so  $A^{-1}$  is symmetric.

(e)  $B^{-1} = -B$  so is still skew symmetric. A  $3 \times 3$  skew symmetric matrix  $B$  is never invertible since  $|B| = 0$ , so it doesn't make sense to see if  $B^{-1}$  is also skew symmetric.

For more details, start a thread on the discussion board.