MATRICES: DETERMINANTS

5 minute review. Remind students how to compute determinants (both 2×2 and 3×3). In the 3×3 case, explain that you can use different rows or columns. As an example, you could show that $|A| = 6$ and $|B| = 3$ for the matrices below.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 13 & 4 \\ 0 & -1 & 1 \\ 3 & 7 & 5 \end{bmatrix}.$$

Class warm-up. With class input, investigate how the sign of the determinant of a 2×2 matrix relates to whether it preserves orientation or not as a map $\mathbb{R}^2 \to \mathbb{R}^2$. Use the matrices on the previous sheet (below), or make up some.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}.$$

Problems. Choose from the below.

1. **A formula for $|AB|$.**
   (a) Let $A = \begin{bmatrix} 3 & 1 \\ -2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix}$. Show that $|AB| = |A| \times |B|$ holds in this case.
   (b) Now let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$. Show that $|AB| = |A| \times |B|$ holds for the general 2×2 case.
   (c) Can you do the general 3×3 case?
2. 3×3 **determinants**. Find the determinants of the following 3×3 matrices by expanding along a row or column of your choice:

$$A = \begin{bmatrix} 5 & 1 & 0 \\ -6 & 0 & 3 \\ 1 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -2 & 9 \\ 1 & 1 & 4 \\ -3 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 5 \\ -2 & 1 & 2 \\ 1 & 7 & 17 \end{bmatrix}, \quad D = \begin{bmatrix} 8 & -3 & -1 \\ 5 & 7 & 2 \\ -2 & 4 & 1 \end{bmatrix}.$$

   Justify why you chose the row or column you did in each case.
3. **Permutation matrices**. A permutation matrix is an $n \times n$ matrix P such that each row and each column has a single 1 and the rest 0's in it. As an example, the identity matrix I_n is a permutation matrix for any choice of n.
   (a) There are exactly two permutation matrices of size 2×2, the identity and one other. Find the other permutation matrix, and its determinant.
   (b) There are exactly six permutation matrices of size 3×3. Find them, and compute all the determinants.
   (c) *Do you notice a pattern in the determinants of permutation matrices?
   (d) *Not all matrices whose entries are only 0 or 1 have determinant ± 1. Find a 3×3 matrix A whose entries are only 0 or 1 such that $|A| = 0$. Can you find a 3×3 matrix B whose entries are only 0 or 1 such that $|B|$ is not $0, 1$ or -1?

Selected answers and hints.

1. $|A| = 23, |B| = -14, |AB| = -322$
2. (a) $|A| = 30$ is best computed using column 2.
   (b) $|B| = 51$ is best computed using row 3.
   (c) $|C| = 0$ is best computed using column 1 (fewest negative signs).
   (d) $|D| = -15$ with no clear choice of best row or column.
3. (a) The other permutation matrix is
   $$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
   and its determinant is -1.
   (b) Of the six 3×3 permutation matrices, half have determinant 1 and half have determinant -1.
   (c) There are $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ permutation matrices of size $n \times n$, half have determinant 1 and half have determinant -1.
   (d)
   $$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
   have determinants $|A| = 0$ and $|B| = -2$.

For more details, start a thread on the discussion board.