MATRICES: MATRIX MULTIPLICATION

5 minute review. Remind students how to multiply matrices (including that in AB, A must be $m \times n$, B must be $n \times p$, and the result is $m \times p$).

Class warm-up. Explain how a 2×2 matrix A gives a function $\mathbb{R}^2 \to \mathbb{R}^2$ using matrix multiplication. "Draw" the functions associated with the matrices below by seeing what each matrix does to a picture in a square of side-length 2. (Any picture you can draw with clear top, bottom, left, and right sides will work; a stick-person style unicorn, perhaps. Then map the square using the matrix, and fill in the picture afterwards, taking care with orientation.)

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}.$$

Problems. Choose from the below.

1. **Matrices and functions**. Let A, B, C, D be the matrices below.
$$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 4 \\ -1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 7 & 0 & 1 \\ 1 & 2 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & 1 \\ -1 & 0 \\ 2 & 4 \end{bmatrix}.$$
Compute AB, AC, BB^T, B^TB, CD and DC.

2. **Orthogonal matrices**. A square matrix is *orthogonal* if $AA^T = I$.
   (a) Show that $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal for any angle θ.
   (b) Find four different 2×2 orthogonal matrices with *integer* entries.
   (c) Is every 2×2 orthogonal matrix of the form A_θ for some angle θ?

3. **Cancellation of matrices**.
   (a) Find two distinct 3×3 matrices A, B such that $AC = BC$ where C is
$$C = \begin{bmatrix} 1 & 3 & -4 \\ 2 & 1 & 7 \\ 1 & -2 & 11 \end{bmatrix}.$$
   (Hint: $AC = BC$ is equivalent to $(A - B)C = 0$.)
   (b) Can you pick A, B such that both $AC = BC$ and $CA = CB$ hold?

4. **The characteristic polynomial**. Let
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}.$$
   (a) Find A^2 and A^3.
   (b) Check that $A^3 - 2A^2 - A = 0$.
   (c) Now find B^2 and B^3.
   (d) Can you find integers r, s such that $B^3 + rB^2 + sB = 0$?

For the warm-up,

- A_1 is the identity function, it doesn't change the square at all.
- A_2 swaps the x and y coordinates; that is, it flips the square across the line $y = x$.
- A_3 sends (x, y) to $(x + 2y, y)$, so turns the square into a parallelogram.
- A_4 sends (x, y) to $(3y - x, x + y)$.

Selected answers and hints.

1.
$$AB = \begin{bmatrix} 6 & 2 \\ 12 & 12 \end{bmatrix}, \quad AC = \begin{bmatrix} 5 & -4 & -9 \\ 21 & 0 & 3 \end{bmatrix}, \quad BB^T = \begin{bmatrix} 32 & 0 \\ 0 & 2 \end{bmatrix},$$
$$B^T B = \begin{bmatrix} 17 & 15 \\ 15 & 17 \end{bmatrix}, \quad CD = \begin{bmatrix} -19 & 11 \\ 5 & 21 \end{bmatrix}, \quad DC = \begin{bmatrix} -20 & 2 & 2 \\ -7 & 0 & -1 \\ 18 & 8 & 22 \end{bmatrix}.$$

2. (a) $AA^T = I$ follows from $\sin^2\theta + \cos^2\theta = 1$.
   (b) The possibilities are $\begin{bmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$.
   (c) Not quite. Every orthogonal matrix is either of the form $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, which represents a rotation as a function $\mathbb{R}^2 \to \mathbb{R}^2$, or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, which represents a reflection.
3. (a) Pick any A, B such that
$$A - B = \begin{bmatrix} \alpha & -\alpha & \alpha \\ \beta & -\beta & \beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$
   for any constants α, β, γ.
   (b) Any A, B for which $A - B$ is a scalar multiple of $\begin{bmatrix} -5 & 5 & -5 \\ 3 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ works.
4. (a)
$$A^2 = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$
$$A^3 = \begin{bmatrix} 7 & 5 \\ 10 & 7 \end{bmatrix}$$
   (b) Easy to verify.
   (c)
$$B^2 = \begin{bmatrix} 13 & -4 \\ -3 & 16 \end{bmatrix}$$
$$B^3 = \begin{bmatrix} 1 & 60 \\ 45 & -44 \end{bmatrix}$$
   (d) $B^3 + B^2 - 14B = 0$

For more details, start a thread on the discussion board.