

MATRICES: BASIC OPERATIONS

5 minute review. Cover the basics of matrices (size, row, columns, (i, j) entry is row i , column j), matrix addition, scalar multiplication, and transposes.

Class warm-up. Let A, B be the 2×2 -matrices below.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 4 \\ -1 & 1 \end{bmatrix}.$$

Compute the transposes, and then find $A+2B^T$, $A^T - B$, $2A - A^T + B^T$ (or similar).

Problems. Choose from the below.

1. **Symmetric and skew-symmetric matrices.** A matrix A is called *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

(a) Write down a general 2×2 symmetric matrix.

(b) Write down a general 2×2 skew-symmetric matrix.

(c) Can you express $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix? What about a general matrix A ? What about the 3×3 case?

2. **Triangular matrices.** An upper triangular 3×3 matrix is one of the form

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}.$$

(a) Express $A = \begin{bmatrix} 1 & 6 & -1 \\ 2 & 0 & 3 \\ 4 & -4 & 4 \end{bmatrix}$ as the sum of an upper triangular matrix and a symmetric matrix. Can you do this in more than one way?

(b) Now find a matrix which is both upper triangular and symmetric, but is not the zero matrix.

(c) Now express the matrix A as the sum of an upper triangular and *skew-symmetric* matrix. Can you do this in more than one way?

(d) Can you find a matrix which is both upper triangular and skew-symmetric?

3. **Linear combinations***

(a) Can you obtain the identity matrix from the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}$$

using only the operations of matrix addition, scalar multiplication, and transpose? In other words, can you find scalars $\alpha, \beta, \gamma, \delta$ such that

$$\alpha A + \beta A^T + \gamma B + \delta B^T = I_2?$$

(b) Can you choose different matrices A, B above that will solve this equation? If you choose A, B “at random,” do you think you can solve this equation or not?

Selected answers and hints.

1. (a)

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$$

(c) For the 2×2 case, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & d \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}(b-c) \\ \frac{1}{2}(c-b) & 0 \end{bmatrix}$.This can be written as $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, which also works for bigger matrices.

2. (a) Many possible answers. One is

$$\begin{bmatrix} 1 & 6 & -1 \\ 2 & 0 & 3 \\ 4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & -4 \\ 4 & -4 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 4 & -5 \\ 0 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) The identity matrix, or any matrix with nonzero entries only along the main diagonal, is symmetric and upper triangular.

(c) Only one possible solution,

$$\begin{bmatrix} 1 & 6 & -1 \\ 2 & 0 & 3 \\ 4 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & 4 \\ 4 & -4 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

(d) The only matrix which is upper triangular and skew-symmetric is the zero matrix. If A is upper triangular, then it has zeroes below the diagonal. If it is skew-symmetric it has them along the diagonal as well, and then $A^T = -A$ means it has them above the diagonal.3. (a) Not possible with the A, B given.

(b) Random choices will not work. Does work with

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

as $I = A - 2B - 3B^T$.

For more details, start a thread on the discussion board.