IMPROPER INTEGRALS

5 minute review. Recap two types of improper integral: (1) integrals over an infinite domain, and (2) integrals with isolated points where the integrand is not defined. Explain how both may be defined using limits:

$$\int_a^\infty f(x)dx = \lim_{X\to\infty} \int_a^X f(x)dx, \qquad \int_a^b g(x)dx = \lim_{\epsilon\to 0^+} \int_a^{c-\epsilon} g(x)dx + \lim_{\epsilon\to 0^+} \int_{c+\epsilon}^b g(x)dx.$$

(Here I'm assuming $g(x)$ is not defined at $x = c$.)

Class warm-up. With class input, decide which of the following are well-defined and finite, and evaluate where possible.

$$\int_0^\infty x dx, \quad \int_0^\infty x^{-2} dx, \quad \int_1^\infty \ln x\, dx, \quad \int_0^1 \ln x\, dx, \quad \int_{-2}^2 |x-1| \ln |x|\, dx.$$

Problems. (Choose from the below)

1. **Which of these integrals are valid?** Evaluate those that are well-defined.

   (a) $\displaystyle\int_1^\infty \frac{1}{x} dx$

   (b) $\displaystyle\int_0^1 \frac{1}{x} dx$

   (c) $\displaystyle\int_1^\infty x^{-1.1} dx$

   (d) $\displaystyle\int_0^1 x^{-0.9} dx$

   (e) $\displaystyle\int_0^1 x \ln(x) dx$

   (f) $\displaystyle\int_0^\infty \sin(x) e^{-x} dx$

   (g) $\displaystyle\int_{-\infty}^0 e^x dx$

   (h) $\displaystyle\int_0^{\pi/2} \tan x dx$

   (i) $\displaystyle\int_{-1}^1 |x| dx$

   (j) $\displaystyle\int_{-1}^2 \frac{1}{x^3} dx$

2. **The Gamma function**. The Gamma function $\Gamma(n)$ is defined by an improper integral: $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$.

   (a) Show that $\Gamma(1) = 1$.

   (b) By integrating by parts, show that $\Gamma(n) = (n-1)\Gamma(n-1)$ for $n > 1$. Hence show that $\Gamma(n) = (n-1)!$ when n is any positive integer.

   (c) Now let n be a real number. For which values of n is $\Gamma(n)$ well-defined according to this definition?

3. **The area under a Gaussian***. The 'Gaussian' function $f(x) = \exp(-x^2)$ appears in many contexts; for example, in the 'normal distribution' ('bell curve') in probability, which arises as the large-N limit of the binomial distribution. Here we consider the total area under a Gaussian,

   $$I = \int_{-\infty}^\infty \exp(-x^2) dx.$$

   (a) Sketch the integrand. Is I well-defined?

   (b) The integral can be found using the following result:

   $$I^2 = \int_0^\infty 2\pi r \exp(-r^2) dr$$

   (ask your tutor where this comes from!). Use the formula above to find I^2 and hence I. Use your new result to show that $\Gamma(1/2) = \sqrt{\pi}$.

For the warm-up, the answers are (in order): undefined [upper limit]; undefined [lower]; undefined [upper], -1, $4\ln 2 - 7/2$.

Selected answers and hints.

1.
   (a) divergent due to upper limit
   (b) divergent due to lower limit
   (c) 10
   (d) 10
   (e) $-1/4$
   (f) $1/2$
   (g) 1
   (h) divergent due to upper limit
   (i) 1
   (j) ill-defined (can't find the limit as $x \to 0$). Could argue by symmetry for $3/8$.
2. (c) For $n > 0$.
3. (a) Yes. (b) $I = \sqrt{\pi}$. To find $\Gamma(1/2)$, let $u = x^2$, and show that $I = 2\int_0^\infty e^{-x^2} dx = \int_0^\infty u^{-1/2} e^{-u} du$. Now compare with the definition of $\Gamma(n)$.

For more details, start a thread on the discussion board.