

DEFINITE INTEGRATION

5 minute review. Briefly recap why $I = \int_a^b f(x)dx = F(b) - F(a)$, where $\frac{dF}{dx} = f$. Split the area under a curve into n equal-width bars of width $h = (b - a)/n$. Show that $I \approx h \times (f(x_0) + f(x_1) + \dots + f(x_{n-1})) + O(nh^2)$, and insert $f(x_i) \approx \frac{F(x_{i+1}) - F(x_i)}{h} + O(h)$. Argue that ‘interior’ terms cancel out, leaving only $F(x_n) - F(x_0)$, where $x_0 = a$ and $x_n = b$. Justify why the error terms disappear in limit $n \rightarrow \infty$.

Class warm-up. Integrate $\int_0^1 xe^{-x} dx$, using integration by parts and $[\]_a^b$ notation.

Problems. (Choose from the below)

1. **Definite integrals.** Evaluate the following definite integrals:

<p>(a) $\int_1^2 \ln x dx$</p> <p>(b) $\int_{1/2}^{3/4} \frac{dt}{(1-t)(1+2t)}$</p> <p>(c) $\int_1^4 \frac{dt}{(1+t)^2}$</p>	<p>(d) $\int_1^{\sqrt{3}} \frac{dx}{1+x^2}$</p> <p>(e) $\int_0^\pi e^{-x} \sin(x) dx$</p> <p>(f) $\int_0^{\ln 2} \sqrt{e^x - 1} dx$ (Hint: let $u = \sqrt{e^x - 1}$.)</p>
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2. **Definite integrals with functions of x in their limits.**

(a) Let $I(x) = \int_x^{x^2} y \ln(y) dy$. By integrating, show that $I(x) = \frac{x^2}{4}(1-x^2) + \frac{x^2}{2}(2x^2 - 1) \ln|x|$. Now by differentiating your answer, show that $\frac{dI}{dx} = (4x^2 - 1)x \ln|x|$.

(b) Apply the following general formula to reach the same result:

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(y) dy = \frac{db}{dx} \times f(b(x)) - \frac{da}{dx} \times f(a(x)).$$

(c) Consider how the region of integration changes when x changes slightly, $x \rightarrow x + \epsilon$. Recall that $a(x + \epsilon) \approx a(x) + \epsilon a'(x)$. Use differentiation from first principles and the approach of Q2 to derive the formula above.

3. **Integration from first principles***. Let $I = \int_0^a e^x dx$. In this question we will show that $I = e^a - 1$ by applying ‘integration from first principles’, using

$$\int f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} f(x_k) h_k, \quad \text{where } h_k = x_{k+1} - x_k.$$

(a) Sketch the area represented by the integral. Now split the area into n equal-width bars. Show that: (i) the width of each bar is $h_k = h = a/n$, and (ii) $x_k = kh$. By inserting $f(x_k) = e^{kh}$ in the first principles formula, show that

$$I = \lim_{n \rightarrow \infty} \frac{a}{n} S_n, \quad \text{where } S_n = \sum_{k=0}^{n-1} e^{ka/n}$$

(b) Show that S_n is a geometric series, $S_n = 1 + b + b^2 + \dots + b^{n-1}$, where $b = e^{a/n}$. Sum the geometric series to show that $S_n = \frac{e^a - 1}{e^{a/n} - 1}$.

(c) By using the Maclaurin series expansion for $e^{a/n}$ and taking the limit $n \rightarrow \infty$, show that $I = e^a - 1$ as expected.

For the warm-up,

$$\int_0^1 xe^{-x} dx = [-xe^{-x}]_0^1 + \int_0^1 e^{-x} dx = [-(x+1)e^{-x}]_0^1 = -2e^{-1} + 1 \times e^0 = 1 - 2e^{-1} \approx 0.264.$$

Selected answers and hints.

1. (a) $2 \ln 2 - 1$, (b) $\frac{1}{3} \ln(5/2)$, (c) $3/10$, (d) $\pi/12$, (e) $\frac{1}{2}(1 + e^{-\pi})$, (f) $2 - \pi/2$

For more details, start a thread on the discussion board.