FURTHER INTEGRATION

5 minute review. Remind students that hyperbolic substitutions can solve integrals which trigonometric substitutions can't, such as $\int \frac{dx}{\sqrt{1+x^2}}$ ($x = \sinh u$), $\int \frac{dx}{\sqrt{x^2-1}}$ ($x = \cosh u$) and $\int \frac{dx}{1-x^2}$ ($x = \tanh u$). Also remind students that there is a trick to integrating rational functions of $\sin x$ and $\cos x$ using the substitution $t = \tan(\frac{x}{2})$, which gives expressions for $\sin x$ and $\cos x$ in terms of t; see Q2.

Class warm-up. Find $\int \sqrt{1+x^2}\,dx$. If more examples are desired, choose something from the below.

Problems. (Choose from the below)

1. **Find the following indefinite integrals:**.
   (a) $\displaystyle\int \sqrt{x^2-9}\,dx$
   (b) $\displaystyle\int \sqrt{25-x^2}\,dx$
   (c) $\displaystyle\int \sqrt{x^2+4x-5}\,dx$
   (d) $\displaystyle\int \left(1-x^2\right)^{3/2}\,dx$
   (e) $\displaystyle\int \frac{\sin x}{\sin x+\cos x}\,dx$
   (f) $\displaystyle\int \frac{\sinh x}{\sinh x+\cosh x}\,dx$
2. **The $t = \tan(x/2)$ substitution**. Consider $I = \int \frac{dx}{\sin x+\cos x}$.
   (a) Use a double-angle formula to show that $\sin(x+\pi/4) = \frac{1}{\sqrt{2}}(\sin x + \cos x)$. Hence use the substitution $y = x + \pi/4$ to write $I = \frac{1}{\sqrt{2}}\int \frac{dy}{\sin y}$.
   (b) Let $t = \tan(y/2)$. Show that (i) $\frac{dy}{dt} = \frac{2}{1+t^2}$ and (ii) $\sin y = \frac{2t}{1+t^2}$.
   (c) Show that $I = \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)\right| + c$.
3. **Integrals of $\sqrt{a^2-x^2}$**. Let $I = \int \sqrt{a^2-x^2}\,dx$ in the region $|x| < |a|$.
   (a) By making the substitution $x = a\sin u$, show that
   $$I = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + c.$$
   (b) Here's another approach. Starting with $I = \int \sqrt{a^2-x^2}\,dx$ and treating it as a function of both x and a, one can show that $\frac{\partial I}{\partial a} = a\sin^{-1}(x/a)$ (how?). Now, integrate with respect to a, treating x as a constant and using integration by parts with $u = \sin^{-1}(x/a)$ and $\frac{dv}{da} = a$.
4. **Consistency of integrals**. Some integrals can be expressed in more than one way, and the results can look superficially rather different. For example, $\int \frac{dx}{1-x^2}$ can be written as both $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) + c$ and $\tanh^{-1}(x) + c$.
   (a) By starting with $\tanh(y) = \frac{e^y-e^{-y}}{e^y+e^{-y}} = x$, show that $\tanh^{-1}x = y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.
   (b) Show that the following expressions are consistent, and find the relationships between c_1 and c_2.
      (i) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x + c_1 = -\cos^{-1}x + c_2$ (where $-1 < x < 1$).
      (ii) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + c_1 = \ln\left(x+\sqrt{x^2-1}\right) + c_2$ (where $x \geq 1$).

For the warm-up, $\int \sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\sinh^{-1} x$ using the substitution $x = \sinh u$ and the identity $\cosh^2 u = \frac{1}{2}(1+\cosh(2u))$.

Selected answers and hints.

1. (a) $\frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2}\cosh^{-1}(x/3)$ (b) $\frac{1}{2}x\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}(x/5)$
   (c) $\frac{1}{2}(x+2)\sqrt{x^2+4x-5} - \frac{9}{2}\cosh^{-1}\left(\frac{1}{3}(x+2)\right)$
   (d) $\frac{1}{4}x\sqrt{1-x^2}\left(\frac{5}{2}-x^2\right) + \frac{3}{8}\sin^{-1} x$ (e) $\frac{x}{2} - \frac{1}{2}\ln|\sin(x)+\cos(x)|$ (f) $\frac{1}{2}x + \frac{1}{4}e^{-2x}$
2. (b)(i) $\frac{dt}{dy} = \frac{1}{2}\sec^2(y/2) = \frac{1}{2}\left(1+\tan^2(y/2)\right) = \frac{1}{2}\left(1+t^2\right)$ so $\frac{dy}{dt} = \frac{2}{1+t^2}$.
   (b)(ii) $\sin(y) = 2\sin(y/2)\cos(y/2) = 2\tan(y/2)/\sec^2(y/2) = 2t/(1+t^2)$.
3. Here's one justification of the given expression for $\frac{\partial I}{\partial a}$. Given that $I = \int \sqrt{a^2-x^2}\,dx$, by definition that means that the derivative of I with respect to x is $\sqrt{a^2-x^2}$. In other words, if we treat I as a function of both x and a, then $\frac{\partial I}{\partial x} = \sqrt{a^2-x^2}$. Now, differentiate partially with respect to a to get $\frac{\partial^2 I}{\partial a \partial x} = \frac{a}{\sqrt{a^2-x^2}}$. Since $\frac{\partial^2 I}{\partial a \partial x}$ will also be the result of differentiating $\frac{\partial I}{\partial a}$ with respect to x keeping a constant, it follows that $\frac{\partial I}{\partial a} = \int \frac{a}{\sqrt{a^2-x^2}}dx = a\sin^{-1}(x/a)$, as claimed.

   Equivalently, one can just differentiate the integrand with respect to a, although without the justification above I'm not sure whether you'd have known that was allowed!
4. (a) $(e^{2y}-1) = x(e^{2y}+1) \Rightarrow (1-x)e^{2y} = 1+x \Rightarrow e^{2y} = \frac{1+x}{1-x} \Rightarrow y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.
   (b) $c_2 = c_1 + \pi/2$ (try alternative substitution $x = \cos u$).
   (c) $c_1 = c_2$. To show equivalence, let $x = \cosh(y) \quad \Rightarrow e^{2y} - 2xe^y + 1 = 0 \quad \Rightarrow e^y = x \pm \sqrt{x^2-1} \quad \Rightarrow y = \cosh^{-1} x = \ln(x \pm \sqrt{x^2-1})$ (and think about domain of $\cosh^{-1} x$).

For more details, start a thread on the discussion board.