

FURTHER INTEGRATION

5 minute review. Remind students that hyperbolic substitutions can solve integrals which trigonometric substitutions can't, such as $\int \frac{dx}{\sqrt{1+x^2}}$ ($x = \sinh u$), $\int \frac{dx}{\sqrt{x^2-1}}$ ($x = \cosh u$) and $\int \frac{dx}{1-x^2}$ ($x = \tanh u$). Also remind students that there is a trick to integrating rational functions of $\sin x$ and $\cos x$ using the substitution $t = \tan(\frac{x}{2})$, which gives expressions for $\sin x$ and $\cos x$ in terms of t ; see Q2.

Class warm-up. Find $\int \sqrt{1+x^2} dx$. If more examples are desired, choose something from the below.

Problems. (Choose from the below)

1. **Find the following indefinite integrals:**

(a) $\int \sqrt{x^2 - 9} dx$

(d) $\int (1 - x^2)^{3/2} dx$

(b) $\int \sqrt{25 - x^2} dx$

(e) $\int \frac{\sin x}{\sin x + \cos x} dx$

(c) $\int \sqrt{x^2 + 4x - 5} dx$

(f) $\int \frac{\sinh x}{\sinh x + \cosh x} dx$

2. **The $t = \tan(x/2)$ substitution.** Consider $I = \int \frac{dx}{\sin x + \cos x}$.

(a) Use a double-angle formula to show that $\sin(x + \pi/4) = \frac{1}{\sqrt{2}}(\sin x + \cos x)$. Hence use the substitution $y = x + \pi/4$ to write $I = \frac{1}{\sqrt{2}} \int \frac{dy}{\sin y}$.

(b) Let $t = \tan(y/2)$. Show that (i) $\frac{dy}{dt} = \frac{2}{1+t^2}$ and (ii) $\sin y = \frac{2t}{1+t^2}$.

(c) Show that $I = \frac{1}{\sqrt{2}} \ln \left| \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) \right| + c$.

3. **Integrals of $\sqrt{a^2 - x^2}$.** Let $I = \int \sqrt{a^2 - x^2} dx$ in the region $|x| < |a|$.

(a) By making the substitution $x = a \sin u$, show that

$$I = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1} \left(\frac{x}{a} \right) + c.$$

(b) Here's another approach. Starting with $I = \int \sqrt{a^2 - x^2} dx$ and treating it as a function of both x and a , one can show that $\frac{\partial I}{\partial a} = a \sin^{-1}(x/a)$ (how?). Now, integrate with respect to a , treating x as a constant and using integration by parts with $u = \sin^{-1}(x/a)$ and $\frac{dv}{da} = a$.

4. **Consistency of integrals.** Some integrals can be expressed in more than one way, and the results can look superficially rather different. For example, $\int \frac{dx}{1-x^2}$ can be written as both $\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) + c$ and $\tanh^{-1}(x) + c$.

(a) By starting with $\tanh(y) = \frac{e^y - e^{-y}}{e^y + e^{-y}} = x$, show that $\tanh^{-1} x = y = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$.

(b) Show that the following expressions are consistent, and find the relationships between c_1 and c_2 .

(i) $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c_1 = -\cos^{-1} x + c_2$ (where $-1 < x < 1$).

(ii) $\int \frac{dx}{\sqrt{x^2-1}} = \cosh^{-1} x + c_1 = \ln(x + \sqrt{x^2-1}) + c_2$ (where $x \geq 1$).

For the warm-up, $\int \sqrt{1+x^2} dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2} \sinh^{-1} x$ using the substitution $x = \sinh u$ and the identity $\cosh^2 u = \frac{1}{2}(1 + \cosh(2u))$.

Selected answers and hints.

1. (a) $\frac{1}{2}x\sqrt{x^2-9} - \frac{9}{2} \cosh^{-1}(x/3)$ (b) $\frac{1}{2}x\sqrt{25-x^2} + \frac{25}{2} \sin^{-1}(x/5)$
 (c) $\frac{1}{2}(x+2)\sqrt{x^2+4x-5} - \frac{9}{2} \cosh^{-1}\left(\frac{1}{3}(x+2)\right)$
 (d) $\frac{1}{4}x\sqrt{1-x^2} \left(\frac{5}{2} - x^2\right) + \frac{3}{8} \sin^{-1} x$ (e) $\frac{x}{2} - \frac{1}{2} \ln |\sin(x) + \cos(x)|$ (f) $\frac{1}{2}x + \frac{1}{4}e^{-2x}$
2. (b)(i) $\frac{dt}{dy} = \frac{1}{2} \sec^2(y/2) = \frac{1}{2} (1 + \tan^2(y/2)) = \frac{1}{2} (1 + t^2)$ so $\frac{dy}{dt} = \frac{2}{1+t^2}$.
 (b)(ii) $\sin(y) = 2 \sin(y/2) \cos(y/2) = 2 \tan(y/2) / \sec^2(y/2) = 2t/(1+t^2)$.
3. Here's one justification of the given expression for $\frac{\partial I}{\partial a}$. Given that $I = \int \sqrt{a^2 - x^2} dx$, by definition that means that the derivative of I with respect to x is $\sqrt{a^2 - x^2}$. In other words, if we treat I as a function of both x and a , then $\frac{\partial I}{\partial x} = \sqrt{a^2 - x^2}$. Now, differentiate partially with respect to a to get $\frac{\partial^2 I}{\partial a \partial x} = \frac{a}{\sqrt{a^2 - x^2}}$. Since $\frac{\partial^2 I}{\partial a \partial x}$ will also be the result of differentiating $\frac{\partial I}{\partial a}$ with respect to x keeping a constant, it follows that $\frac{\partial I}{\partial a} = \int \frac{a}{\sqrt{a^2 - x^2}} dx = a \sin^{-1}(x/a)$, as claimed.

 Equivalently, one can just differentiate the integrand with respect to a , although without the justification above I'm not sure whether you'd have known that was allowed!
4. (a) $(e^{2y} - 1) = x(e^{2y} + 1) \Rightarrow (1-x)e^{2y} = 1+x \Rightarrow e^{2y} = \frac{1+x}{1-x} \Rightarrow y = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$.
 (b) $c_2 = c_1 + \pi/2$ (try alternative substitution $x = \cos u$).
 (c) $c_1 = c_2$. To show equivalence, let $x = \cosh(y) \Rightarrow e^{2y} - 2xe^y + 1 = 0 \Rightarrow e^y = x \pm \sqrt{x^2 - 1} \Rightarrow y = \cosh^{-1} x = \ln(x \pm \sqrt{x^2 - 1})$ (and think about domain of $\cosh^{-1} x$).

For more details, start a thread on the discussion board.