

PARTIAL FRACTIONS & INTEGRATION BY PARTS

5 minute review. Recap how *partial fractions* can be used in integration, perhaps with the example $\int \frac{1}{1-x^2} dx$. Recall the product rule for differentiation, $\frac{d}{dx}(uv) = u'v + uv'$, and integrate and rearrange to obtain the *integration by parts* formula

$$\int uv' dx = uv - \int u'v dx.$$

Class warm-up. Find

- (a) $\int \frac{4}{(x-1)(x+1)^2} dx$ (using partial fractions);
 (b) $\int x \ln x dx$ (using integration by parts).

Problems. (Choose from the below)

1. **Partial fractions.** Find the following indefinite integrals using the method of partial fractions as appropriate.

(a) $\int \frac{(x+1)}{x(x+3)} dx$	(d) $\int \frac{y dy}{y^3 - y^2 + y - 1}$
(b) $\int \frac{dx}{x^2 + 3x - 10}$	(e) $\int \frac{dz}{(z-2)^2(z+1)}$
(c) $\int \frac{dx}{(x^2+1)(x+1)}$	(f) $\int \frac{3x+3}{(x-1)^3(2x+1)} dx$

2. **Integration by parts.** Evaluate the following using integration by parts.

(a) $\int te^t dt$	(e) $\int y^3 e^{-y^2} dy$
(b) $\int x^2 \cosh x dx$	(f) $\int \ln(t^2 + a^2) dt$
(c) $\int \ln x dx$	(g) $\int \cosh^{-1} u du$
(d) $\int x^n \ln x dx \quad (n \neq -1)$	(h) $\int \tan^{-1} u du$

3. **Recurrence formulae***

- (a) Let $I_n = \int x^n e^{ax} dx$ where $n \geq 0$ is an integer and a is a (possibly complex) constant. Using integration by parts, show that, for $n > 0$,

$$I_n = \frac{1}{a} (x^n e^{ax} - nI_{n-1})$$

and $I_0 = \frac{1}{a} e^{ax} + c$. Find I_1 , I_2 and I_3 . Show that

$$\frac{I_n}{n!} = \frac{1}{a} \left(\frac{x^n}{n!} e^{ax} - \frac{I_{n-1}}{(n-1)!} \right),$$

and find a general expression for I_n .

- (b) Let $C_n = \int x^n \cos x dx$ and $S_n = \int x^n \sin x dx$. Show that, for $n > 0$,

$$\begin{aligned} C_n &= x^n \sin x - nS_{n-1}, \\ S_n &= -x^n \cos x + nC_{n-1}. \end{aligned}$$

and $C_0 = \sin x + c$, $S_0 = -\cos x + c$. Following a similar approach to above, can you find a general expression for C_n and S_n ?

- (c) When $a = i$, what's the relationship between I_n , S_n and C_n ?

For the warm-up, (a) $\frac{4}{(x-1)(x+1)^2} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{(x+1)^2}$, so $\int \frac{4dx}{(x-1)(x+1)^2} = \ln \left| \frac{x-1}{x+1} \right| + \frac{2}{x+1} + c$; (b) $\int x \ln x dx = \frac{1}{4}x^2(2 \ln x - 1)$.

Selected answers and hints. (All answers should include a constant of integration.)

- (a) $\frac{1}{3} \ln |x(x+3)^2|$, (b) $\frac{1}{7} \ln \left| \frac{x-2}{x+5} \right|$, (c) $\frac{1}{2} \ln |x+1| - \frac{1}{4} \ln |1+x^2| + \frac{1}{2} \tan^{-1}(x)$ (d) $\frac{1}{2} \ln |y-1| - \frac{1}{4} \ln |1+y^2| + \frac{1}{2} \tan^{-1}(y)$, (e) $\frac{1}{9} \ln \left| \frac{z+1}{z-2} \right| - \frac{1}{3(z-2)}$, (f) $\frac{2}{9} \ln \left| \frac{x-1}{2x+1} \right| + \frac{1}{3(x-1)} - \frac{1}{(x-1)^2}$.
- (a) $(t-1)e^t$, (b) $(x^2+2) \sinh x - 2x \cosh x$, (c) $x(\ln x - 1)$ using $u = \ln x$, $v' = 1$, (d) $\frac{x^{n+1}}{(n+1)^2} ((n+1) \ln x - 1)$, (e) $-\frac{1}{2} (y^2 + 1) \exp(-y^2)$, (f) $t \ln(t^2 + a^2) - 2t + 2a \tan^{-1}(t/a)$, (g) $u \cosh^{-1} u - \sqrt{u^2 - 1}$, (h) $u \tan^{-1} u - \frac{1}{2} \ln |1 + u^2|$.
- (a) A general formula for I_n is

$$\begin{aligned} I_n &= n! \frac{e^{ax}}{a^{n+1}} \left(\frac{(ax)^n}{n!} - \frac{(ax)^{n-1}}{(n-1)!} + \frac{(ax)^{n-2}}{(n-2)!} - \dots + (-1)^{n-1} ax + (-1)^n \right) + c \\ &= n! \frac{e^{ax}}{a^{n+1}} \sum_{k=0}^n \frac{(-1)^k (ax)^{n-k}}{(n-k)!} + c. \end{aligned}$$

- (b) The general formulas are

$$\begin{aligned} C_n &= n! \left(\frac{x^n}{n!} - \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-4}}{(n-4)!} - \dots \right) \sin x \\ &\quad + n! \left(\frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-5}}{(n-5)!} - \dots \right) \cos x + c \end{aligned}$$

and

$$\begin{aligned} S_n &= n! \left(\frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-5}}{(n-5)!} - \dots \right) \sin x \\ &\quad - n! \left(\frac{x^n}{n!} - \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-4}}{(n-4)!} - \dots \right) \cos x + c \end{aligned}$$

where the series in each bracket terminates before the power of x involved becomes negative.

- (c) Using Euler's relation, $e^{ix} = \cos x + i \sin x$, we find that $I_n = C_n + iS_n$.

For more details, start a thread on the discussion board.