PARTIAL FRACTIONS & INTEGRATION BY PARTS

5 minute review. Recap how *partial fractions* can be used in integration, perhaps with the example $\int \frac{1}{1-x^2}\,dx$. Recall the product rule for differentiation, $\frac{d}{dx}(uv) = u'v + uv'$, and integrate and rearrange to obtain the *integration by parts* formula

$$\int uv'\,dx = uv - \int u'v\,dx.$$

Class warm-up. Find

(a) $\int \frac{4}{(x-1)(x+1)^2}\,dx$ (using partial fractions);

(b) $\int x \ln x\,dx$ (using integration by parts).

Problems. (Choose from the below)

1. **Partial fractions**. Find the following indefinite integrals using the method of partial fractions as appropriate.

   (a) $\displaystyle\int \frac{(x+1)}{x(x+3)}\,dx$

   (b) $\displaystyle\int \frac{dx}{x^2+3x-10}$

   (c) $\displaystyle\int \frac{dx}{(x^2+1)(x+1)}$

   (d) $\displaystyle\int \frac{y\,dy}{y^3-y^2+y-1}$

   (e) $\displaystyle\int \frac{dz}{(z-2)^2(z+1)}$

   (f) $\displaystyle\int \frac{3x+3}{(x-1)^3(2x+1)}\,dx$

2. **Integration by parts**. Evaluate the following using integration by parts.

   (a) $\displaystyle\int te^t\,dt$

   (b) $\displaystyle\int x^2\cosh x\,dx$

   (c) $\displaystyle\int \ln x\,dx$

   (d) $\displaystyle\int x^n \ln x\,dx \qquad (n \neq -1)$

   (e) $\displaystyle\int y^3 e^{-y^2}\,dy$

   (f) $\displaystyle\int \ln(t^2+a^2)\,dt$

   (g) $\displaystyle\int \cosh^{-1} u\,du$

   (h) $\displaystyle\int \tan^{-1} u\,du$

3. **Recurrence formulae***.

   (a) Let $I_n = \int x^n e^{ax} dx$ where $n \geq 0$ is an integer and a is a (possibly complex) constant. Using integration by parts, show that, for $n > 0$,
   $$I_n = \frac{1}{a}\left(x^n e^{ax} - nI_{n-1}\right)$$
   and $I_0 = \frac{1}{a}e^{ax} + c$. Find I_1, I_2 and I_3. Show that
   $$\frac{I_n}{n!} = \frac{1}{a}\left(\frac{x^n}{n!}e^{ax} - \frac{I_{n-1}}{(n-1)!}\right),$$
   and find a general expression for I_n.

   (b) Let $C_n = \int x^n \cos x\,dx$ and $S_n = \int x^n \sin x\,dx$. Show that, for $n > 0$,
   $$\begin{aligned} C_n &= x^n \sin x - nS_{n-1}, \\ S_n &= -x^n \cos x + nC_{n-1}. \end{aligned}$$
   and $C_0 = \sin x + c$, $S_0 = -\cos x + c$. Following a similar approach to above, can you find a general expression for C_n and S_n?

   (c) When $a = i$, what's the relationship between I_n, S_n and C_n?

For the warm-up, (a) $\frac{4}{(x-1)(x+1)^2} = \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{(x+1)^2}$, so $\int \frac{4\,dx}{(x-1)(x+1)^2} = \ln\left|\frac{x-1}{x+1}\right| + \frac{2}{x+1} + c$; (b) $\int x \ln x \, dx = \frac{1}{4}x^2 (2\ln x - 1)$.

Selected answers and hints. (All answers should include a constant of integration.)

1. (a) $\frac{1}{3}\ln\left|x(x+3)^2\right|$, (b) $\frac{1}{7}\ln\left|\frac{x-2}{x+5}\right|$, (c) $\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|1+x^2| + \frac{1}{2}\tan^{-1}(x)$ (d) $\frac{1}{2}\ln|y-1| - \frac{1}{4}\ln|1+y^2| + \frac{1}{2}\tan^{-1}(y)$, (e) $\frac{1}{9}\ln\left|\frac{z+1}{z-2}\right| - \frac{1}{3(z-2)}$, (f) $\frac{2}{9}\ln\left|\frac{x-1}{2x+1}\right| + \frac{1}{3(x-1)} - \frac{1}{(x-1)^2}$.
2. (a) $(t-1)e^t$, (b) $(x^2+2)\sinh x - 2x\cosh x$, (c) $x(\ln x - 1)$ using $u = \ln x$, $v' = 1$, (d) $\frac{x^{n+1}}{(n+1)^2}\left((n+1)\ln x - 1\right)$, (e) $-\frac{1}{2}\left(y^2+1\right)\exp(-y^2)$, (f) $t\ln(t^2+a^2) - 2t + 2a\tan^{-1}(t/a)$, (g) $u\cosh^{-1}u - \sqrt{u^2-1}$, (h) $u\tan^{-1}u - \frac{1}{2}\ln|1+u^2|$.
3. (a) A general formula for I_n is

$$\begin{aligned} I_n &= n!\frac{e^{ax}}{a^{n+1}}\left(\frac{(ax)^n}{n!} - \frac{(ax)^{n-1}}{(n-1)!} + \frac{(ax)^{n-2}}{(n-2)!} - \ldots + (-1)^{n-1}ax + (-1)^n\right) + c \\ &= n!\frac{e^{ax}}{a^{n+1}}\sum_{k=0}^{n}\frac{(-1)^k(ax)^{n-k}}{(n-k)!} + c. \end{aligned}$$

   (b) The general formulas are

$$\begin{aligned} C_n &= n!\left(\frac{x^n}{n!} - \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-4}}{(n-4)!} - \ldots\right)\sin x \\ &\quad + n!\left(\frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-5}}{(n-5)!} - \ldots\right)\cos x + c \end{aligned}$$

   and

$$\begin{aligned} S_n &= n!\left(\frac{x^{n-1}}{(n-1)!} - \frac{x^{n-3}}{(n-3)!} + \frac{x^{n-5}}{(n-5)!} - \ldots\right)\sin x \\ &\quad - n!\left(\frac{x^n}{n!} - \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-4}}{(n-4)!} - \ldots\right)\cos x + c \end{aligned}$$

   where the series in each bracket terminates before the power of x involved becomes negative.

   (c) Using Euler's relation, $e^{ix} = \cos x + i\sin x$, we find that $I_n = C_n + iS_n$.

For more details, start a thread on the discussion board.