INTEGRATION BY SUBSTITUTION

5 minute review. Recall the chain rule for differentiating a function of a function, namely $\frac{d}{dx}f(u(x)) = \frac{df}{du}\frac{du}{dx}$, then demonstrate and ask for suggestions on integration by substitution using the warm-up examples below, running through these steps: (1) introduce a new variable $u = u(x)$ (or $x = x(u)$); (2) replace the measure using $dx = \frac{dx}{du}du$ (or $dx = \frac{1}{\frac{du}{dx}}du$); (3) integrate with respect to u; (4) replace u with x in the answer.

Class warm-up. Integrate (a) $\int x^2 \exp(x^3)dx$, (b) $\int \frac{x^2}{1+x^3}dx$ and/or (c) $\int \frac{dx}{\sqrt{1-x^2}}$.

Problems. (Choose from the below)

1. **Simple substitutions**. Find the following indefinite integrals by choosing a suitable substitution.

   (a) $\displaystyle\int t\cos(t^2-1)\,dt$
   (b) $\displaystyle\int \sin^2 x\cos x\,dx$
   (c) $\displaystyle\int x\sqrt{1+x^2}\,dx$
   (d) $\displaystyle\int (t+1)\sqrt{t^2+2t}\,dt$
   (e) $\displaystyle\int \frac{x}{x^2+1}\,dx$
   (f) $\displaystyle\int \frac{3x}{x^2+a^2}\,dx$
   (g) $\displaystyle\int \frac{\cos t}{\sqrt{1+\sin t}}\,dt$
   (h) $\displaystyle\int \frac{\cosh u}{1+\sinh^2 u}\,du$

2. **Trigonometric substitutions**. Find the indefinite integrals using the suggested substitutions.

   (a) $\displaystyle\int \frac{1}{1+9x^2}\,dx, \quad 3x = \tan u;$
   (b) $\displaystyle\int \frac{1}{\sqrt{25-16x^2}}\,dx, \quad 4x = 5\sin u;$
   (c) $\displaystyle\int \tan x\,dx, \quad u = \cos x;$
   (d) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx, \quad x+1 = 2\tan u;$
   (e) $\displaystyle\int \sec 2x\tan 2x\,dx, \quad u = \cos 2x.$

3. **Completing the square**. Find the following.

   (a) $\displaystyle\int \frac{dx}{x^2+6x+10}$
   (b) $\displaystyle\int \frac{dx}{\sqrt{3-2x-x^2}}$
   (c) $\displaystyle\int \frac{\cos x\,dx}{\sin^2 x+2\sin x+10}$
   (d) $\displaystyle\int \frac{dx}{x^2+10x+29}$.

4. **Log-of-a-log***. Use the substitution $x = e^u$ to show that

$$\int \frac{dx}{x\ln x} = \ln(\ln x) + c.$$

   Let $I_n(x)$ be defined as follows.

$$I_1(x) = \int \frac{dx}{x\ln x},\ I_2(x) = \int \frac{dx}{x\ln(x)\ln(\ln x)},\ I_3(x) = \int \frac{dx}{x\ln(x)\ln(\ln x)\ln(\ln(\ln x))},\ldots$$

   Note that, as n gets larger, x has to be very large and positive in order that its logarithm can be taken repeatedly.

   Find $I_2(x)$ using integration by substitution. Show that $I_n(e^u) = I_{n-1}(u)$ (in other words, that substituting $x = e^u$ in $I_n(x)$ gives $I_{n-1}(u)$), and hence write down a general formula for $I_n(x)$. Check it is valid by differentiating.

For the warm-up, $\int x^2\exp(x^3)dx = \frac{1}{3}\exp(x^3)$, (b) $\int \frac{x^2}{1+x^3}dx = \frac{1}{3}\ln(1+x^3)$ and (c) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$.

Selected answers and hints. (All answers should include a constant of integration.)

1. (a) $\frac{1}{2}\sin(t^2-1)$, (b) $\frac{1}{3}\sin^3 x$, (c) $\frac{1}{3}\left(1+x^2\right)^{3/2}$, (d) $\frac{1}{3}(t^2+2t)^{3/2}$ (e) $\frac{1}{2}\ln(1+x^2)$, (f) $\frac{3}{2}\ln\left(x^2+a^2\right)$, (g) $2\sqrt{1+\sin t}$, (h) $\tan^{-1}(\sinh u)$.
2. (a) $\frac{1}{3}\arctan(3x)$, (b) $\frac{1}{4}\arcsin\left(\frac{4}{5}x\right)$, (c) $\ln|\sec x|$, (d) $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+1)\right)$, (e) $\frac{1}{2}\sec(2x)$.
3. (a) $\tan^{-1}(x+3)$, (b) $\sin^{-1}\left(\frac{1}{2}(x+1)\right)$, (c) $\frac{1}{3}\tan^{-1}\left(\frac{1}{3}(\sin x+1)\right)$ (d) $\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+5)\right)$.
4. $I_n(e^u)$ is the result of making the substitution $x = e^u$ in $I_n(x)$. That is,

$$\begin{aligned} I_n(e^u) &= \int \frac{dx}{e^u \ln(e^u)\ldots\ln(\ln\ldots(\ln e^u)\ldots)} \\ &= \int \frac{\frac{dx}{du}du}{e^u u\ldots\ln(\ln\ldots(u)\ldots)} \\ &= \int \frac{e^u du}{e^u u\ldots\ln(\ln\ldots(u)\ldots)} \\ &= I_{n-1}(u). \end{aligned}$$

   Putting $x = \ln u$ in the given identity, we get

$$I_n(x) = I_{n-1}(\ln x) = I_{n-2}(\ln(\ln x)) = \ldots$$

   It follows that

$$I_n(x) = \underbrace{\ln(\ln(\ln\ldots(\ln(x))\ldots))}_{n+1 \text{ times}} + c.$$

For more details, start a thread on the discussion board.