

INTEGRATION BY SUBSTITUTION

5 minute review. Recall the chain rule for differentiating a function of a function, namely $\frac{d}{dx}f(u(x)) = \frac{df}{du} \frac{du}{dx}$, then demonstrate and ask for suggestions on integration by substitution using the warm-up examples below, running through these steps: (1) introduce a new variable $u = u(x)$ (or $x = x(u)$); (2) replace the measure using $dx = \frac{dx}{du} du$ (or $dx = \frac{1}{\frac{du}{dx}} du$); (3) integrate with respect to u ; (4) replace u with x in the answer.

Class warm-up. Integrate (a) $\int x^2 \exp(x^3) dx$, (b) $\int \frac{x^2}{1+x^3} dx$ and/or (c) $\int \frac{dx}{\sqrt{1-x^2}}$.

Problems. (Choose from the below)

1. **Simple substitutions.** Find the following indefinite integrals by choosing a suitable substitution.

(a) $\int t \cos(t^2 - 1) dt$	(e) $\int \frac{x}{x^2 + 1} dx$
(b) $\int \sin^2 x \cos x dx$	(f) $\int \frac{3x}{x^2 + a^2} dx$
(c) $\int x \sqrt{1 + x^2} dx$	(g) $\int \frac{\cos t}{\sqrt{1 + \sin t}} dt$
(d) $\int (t + 1) \sqrt{t^2 + 2t} dt$	(h) $\int \frac{\cosh u}{1 + \sinh^2 u} du$

2. **Trigonometric substitutions.** Find the indefinite integrals using the suggested substitutions.

(a) $\int \frac{1}{1 + 9x^2} dx, \quad 3x = \tan u;$	(d) $\int \frac{1}{x^2 + 2x + 5} dx, \quad x + 1 = 2 \tan u;$
(b) $\int \frac{1}{\sqrt{25 - 16x^2}} dx, \quad 4x = 5 \sin u;$	(e) $\int \sec 2x \tan 2x dx, \quad u = \cos 2x.$
(c) $\int \tan x dx, \quad u = \cos x;$	

3. **Completing the square.** Find the following.

(a) $\int \frac{dx}{x^2 + 6x + 10}$	(c) $\int \frac{\cos x dx}{\sin^2 x + 2 \sin x + 10}$
(b) $\int \frac{dx}{\sqrt{3 - 2x - x^2}}$	(d) $\int \frac{dx}{x^2 + 10x + 29}.$

4. **Log-of-a-log***. Use the substitution $x = e^u$ to show that

$$\int \frac{dx}{x \ln x} = \ln(\ln x) + c.$$

Let $I_n(x)$ be defined as follows.

$$I_1(x) = \int \frac{dx}{x \ln x}, \quad I_2(x) = \int \frac{dx}{x \ln(x) \ln(\ln x)}, \quad I_3(x) = \int \frac{dx}{x \ln(x) \ln(\ln x) \ln(\ln(\ln x))}, \dots$$

Note that, as n gets larger, x has to be very large and positive in order that its logarithm can be taken repeatedly.

Find $I_2(x)$ using integration by substitution. Show that $I_n(e^u) = I_{n-1}(u)$ (in other words, that substituting $x = e^u$ in $I_n(x)$ gives $I_{n-1}(u)$), and hence write down a general formula for $I_n(x)$. Check it is valid by differentiating.

For the warm-up, $\int x^2 \exp(x^3) dx = \frac{1}{3} \exp(x^3)$, (b) $\int \frac{x^2}{1+x^3} dx = \frac{1}{3} \ln(1+x^3)$ and (c) $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x$.

Selected answers and hints. (All answers should include a constant of integration.)

- (a) $\frac{1}{2} \sin(t^2-1)$, (b) $\frac{1}{3} \sin^3 x$, (c) $\frac{1}{3} (1+x^2)^{3/2}$, (d) $\frac{1}{3} (t^2+2t)^{3/2}$ (e) $\frac{1}{2} \ln(1+x^2)$,
(f) $\frac{3}{2} \ln(x^2+a^2)$, (g) $2\sqrt{1+\sin t}$, (h) $\tan^{-1}(\sinh u)$.
- (a) $\frac{1}{3} \arctan(3x)$, (b) $\frac{1}{4} \arcsin(\frac{4}{5}x)$, (c) $\ln|\sec x|$, (d) $\frac{1}{2} \tan^{-1}(\frac{1}{2}(x+1))$, (e) $\frac{1}{2} \sec(2x)$.
- (a) $\tan^{-1}(x+3)$, (b) $\sin^{-1}(\frac{1}{2}(x+1))$, (c) $\frac{1}{3} \tan^{-1}(\frac{1}{3}(\sin x+1))$ (d) $\frac{1}{2} \tan^{-1}(\frac{1}{2}(x+5))$.
- $I_n(e^u)$ is the result of making the substitution $x = e^u$ in $I_n(x)$. That is,

$$\begin{aligned} I_n(e^u) &= \int \frac{dx}{e^u \ln(e^u) \dots \ln(\ln \dots (\ln e^u) \dots)} \\ &= \int \frac{\frac{dx}{du} du}{e^u u \dots \ln(\ln \dots (u) \dots)} \\ &= \int \frac{e^u du}{e^u u \dots \ln(\ln \dots (u) \dots)} \\ &= I_{n-1}(u). \end{aligned}$$

Putting $x = \ln u$ in the given identity, we get

$$I_n(x) = I_{n-1}(\ln x) = I_{n-2}(\ln(\ln x)) = \dots$$

It follows that

$$I_n(x) = \underbrace{\ln(\ln(\ln \dots (\ln(x)) \dots))}_{n+1 \text{ times}} + c.$$

For more details, start a thread on the discussion board.