

INTEGRATION BY INSPECTION

5 minute review. Remind students that $F(x)$ is an *indefinite integral* of $f(x)$ if and only if $F'(x) = f(x)$, and that we write

$$\int f(x) dx = F(x) + c.$$

Here $f(x)$ is the *integrand* and dx is the *measure of integration*. The indefinite integral is only defined up to an additive constant since, for any indefinite integral $F(x)$, $\frac{d}{dx}(F+c) = \frac{dF}{dx} + \frac{d}{dx}(c) = f + 0 = f$. Describe the process of integration by inspection.

Class warm-up. “Every function that can be differentiated generates a function that can be integrated.” (a) Find the derivative of $\exp(-x^3)$, and hence find $\int x^2 \exp(-x^3) dx$. (b) Find the derivatives of e^{-x} and xe^{-x} . By taking a linear combination of results, find $\int xe^{-x} dx$. (You could mention that (a) can be done with integration by substitution, and (b) with integration by parts, which will follow next week.)

Problems. (Choose from the below)

1. **Integration by inspection.** By looking for functions F such that $F'(x) = f(x)$, find the indefinite integrals $\int f(x) dx$ for the following integrands.

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|-------------------------|--------------------------|----------------------------|
| (a) $x^2 + x^3$ | (e) $\frac{1}{(3x+2)^3}$ | (j) $(1-x^2)/(1+x)$ |
| (b) $(1+x)^2$ | (f) e^{4x} | (k) $\frac{1}{x}$ [see Q4] |
| (c) $\frac{1}{x^2}$ | (g) $\sin 3x$ | (l) $\frac{1}{2x+1}$ |
| (d) $\frac{1}{(x+1)^2}$ | (h) $\cosh 4x$ | (m) $\frac{x}{1+x}$ |
| | (i) $\sec^2 x$ | |

2. **Integrating powers of trigonometric functions.** By using an appropriate double-angle formula and the identity $\sin^2 x + \cos^2 x = 1$, find

- (a) $\int \cos^2 x dx$;
 (b) $\int \sin^3 x dx$;
 (c) $\int \sin^4 x dx$;
 (d) Can you find $\int \cos^4 x dx$ using (c)? (Hint: replace x with $x + \pi/2$. It's possible to do this without using integration by substitution!)

3. **Other integrands.** Let $F(x)$ be an indefinite integral of $f(x)$, where

$$f(x) = x(x-1)^2(x-2)^3.$$

- (a) Find and classify the stationary points of $F(x)$.
 (b) Sketch $F(x)$ in the region $0 \leq x \leq 2$, assuming that $F(0) = 0$.

4. **Logarithms***.

- (a) Let $y = \ln(x)$, where $x > 0$. By taking exponentials of both sides and differentiating, show that $y' = \frac{1}{x}$. Hence write down the indefinite integral of $\frac{1}{x}$ for $x > 0$.
 (b) Repeat with $y = \ln(-x)$, where $x < 0$.
 (c) What's the best expression for $\int \frac{1}{x} dx$?
 (d) Now repeat for $y = \ln(g(x))$ and $\ln(-g(x))$ for an unknown function $g(x)$. Hence find an expression for $\int \frac{g'(x)}{g(x)} dx$.

For the warm-up, $\int x^2 \exp(-x^3) dx = -\frac{1}{3} \exp(-x^3)$ and $\int x e^{-x} dx = -e^{-x}(x+1)$.

Selected answers and hints.

1. All answers should include a constant of integration, omitted here.

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|---|------------------------------|------------------------------|
| (a) $\frac{1}{3}x^3 + \frac{1}{4}x^4$; | (e) $-\frac{1}{6(3x+2)^2}$; | (j) $x - \frac{1}{2}x^2$; |
| (b) $\frac{1}{3}(1+x)^3$; | (f) $\frac{1}{4}e^{4x}$; | (k) $\ln x $; |
| (c) $-x^{-1}$; | (g) $-\frac{1}{3}\cos 3x$; | (l) $\frac{1}{2}\ln 2x+1 $; |
| (d) $-(x+1)^{-1}$; | (h) $\frac{1}{4}\sinh 4x$; | (m) $x - \ln 1+x $. |
| | (i) $\tan x$; | |

2. (a) $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$;

(b) $-\cos x + \frac{1}{3}\cos^3 x + c$ (or equivalents);

(c) $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{8}(3 + 2\sin^2 x)\cos x \sin x + c$.

(d) Let $F(x) = \frac{3}{8}x - \frac{1}{8}(3 + 2\sin^2 x)\cos x \sin x$ and let $u = x + \pi/2$. Then, using the fact that $\sin(x + \pi/2) = \cos x$ and $\cos(x + \pi/2) = -\sin x$,

$$\begin{aligned} F(u) &= \frac{3}{8}u - \frac{1}{8}(3 + 2\sin^2 u)\cos u \sin u \\ &= \frac{3}{8}x + \frac{3}{8}(\pi/2) - \frac{1}{8}(3 + 2\cos^2(x))(-\sin(x))\cos(x) \\ &= \frac{3}{8}x + \frac{1}{8}(3 + 2\cos^2 x)\sin x \cos x + \frac{3\pi}{16}. \end{aligned}$$

But $F(x)$ differentiates to give $\sin^4 x$, so differentiating the above we get

$$\frac{d}{dx}(F(u)) = \frac{d}{du}(F(u)) \cdot \frac{du}{dx} = F'(u) \cdot 1 = \sin^4(u) = \cos^4 x.$$

It follows that the expression for $F(u)$ above is an indefinite integral of $\cos^4 x$; in other words, $\int \cos^4 x dx = \frac{3}{8}x + \frac{1}{8}(3 + 2\cos^2 x)\sin x \cos x + c$.

3. $F(x)$ has stationary points at $x = 0, 1$, and 2 , which are maximum, inflexion and minimum, respectively. $F(x)$ passes through zero at $x = 0$. Hopefully that's enough info to draw a rough sketch.

4. (a) $\int \frac{1}{x} dx$ seems to be $\ln(x) + C$.

(b) Now $\int \frac{1}{x} dx$ seems to be $\ln(-x) + C$.

(c) The above is best summarised by $\int \frac{1}{x} dx = \ln(|x|) + C$, as I'm sure you already know.

(d) Similarly, we find $\int \frac{g'(x)}{g(x)} dx = \ln(|g(x)|) + C$.

For more details, start a thread on the discussion board.