# INTEGRATION BY INSPECTION

**5 minute review.** Remind students that $F(x)$ is an *indefinite integral* of $f(x)$ if and only if $F'(x) = f(x)$, and that we write

$$\int f(x)\,dx = F(x) + c.$$

Here $f(x)$ is the *integrand* and $dx$ is the *measure of integration*. The indefinite integral is only defined up to an additive constant since, for any indefinite integral $F(x)$, $\frac{d}{dx}(F+c) = \frac{dF}{dx} + \frac{d}{dx}(c) = f + 0 = f$. Describe the process of integration by inspection.

**Class warm-up.** "Every function that can be differentiated generates a function that can be integrated." (a) Find the derivative of $\exp(-x^3)$, and hence find $\int x^2 \exp(-x^3)\,dx$. (b) Find the derivatives of $e^{-x}$ and $xe^{-x}$. By taking a linear combination of results, find $\int xe^{-x}\,dx$. (You could mention that (a) can be done with integration by substitution, and (b) with integration by parts, which will follow next week.)

**Problems.** (Choose from the below)

1. **Integration by inspection**. By looking for functions $F$ such that $F'(x) = f(x)$, find the indefinite integrals $\int f(x)\,dx$ for the following integrands.

   (a) $x^2 + x^3$
   (b) $(1+x)^2$
   (c) $\dfrac{1}{x^2}$
   (d) $\dfrac{1}{(x+1)^2}$
   (e) $\dfrac{1}{(3x+2)^3}$
   (f) $e^{4x}$
   (g) $\sin 3x$
   (h) $\cosh 4x$
   (i) $\sec^2 x$
   (j) $(1-x^2)/(1+x)$
   (k) $\dfrac{1}{x}$ [see Q4]
   (l) $\dfrac{1}{2x+1}$
   (m) $\dfrac{x}{1+x}$

2. **Integrating powers of trigonometric functions**. By using an appropriate double-angle formula and the identity $\sin^2 x + \cos^2 x = 1$, find

   (a) $\int \cos^2 x\,dx$;

   (b) $\int \sin^3 x\,dx$;

   (c) $\int \sin^4 x\,dx$;

   (d) Can you find $\int \cos^4 x\,dx$ using (c)? (Hint: replace $x$ with $x + \pi/2$. It's possible to do this without using integration by substitution!)

3. **Other integrands**. Let $F(x)$ be an indefinite integral of $f(x)$, where

   $$f(x) = x(x-1)^2(x-2)^3.$$

   (a) Find and classify the stationary points of $F(x)$.

   (b) Sketch $F(x)$ in the region $0 \le x \le 2$, assuming that $F(0) = 0$.

4. **Logarithms**$^\star$.

   (a) Let $y = \ln(x)$, where $x > 0$. By taking exponentials of both sides and differentiating, show that $y' = \frac{1}{x}$. Hence write down the indefinite integral of $\frac{1}{x}$ for $x > 0$.

   (b) Repeat with $y = \ln(-x)$, where $x < 0$.

   (c) What's the best expression for $\int \frac{1}{x}\,dx$?

   (d) Now repeat for $y = \ln(g(x))$ and $\ln(-g(x))$ for an unknown function $g(x)$. Hence find an expression for $\int \frac{g'(x)}{g(x)}\,dx$.

For the warm-up, $\int x^2 \exp(-x^3)\,dx = -\frac{1}{3}\exp(-x^3)$ and $\int xe^{-x}\,dx = -e^{-x}(x+1)$.

**Selected answers and hints.**

1. All answers should include a constant of integration, omitted here.
   (a) $\frac{1}{3}x^3 + \frac{1}{4}x^4$;
   (b) $\frac{1}{3}(1+x)^3$;
   (c) $-x^{-1}$;
   (d) $-(x+1)^{-1}$;
   (e) $-\frac{1}{6(3x+2)^2}$;
   (f) $\frac{1}{4}e^{4x}$;
   (g) $-\frac{1}{3}\cos 3x$;
   (h) $\frac{1}{4}\sinh 4x$;
   (i) $\tan x$;
   (j) $x - \frac{1}{2}x^2$;
   (k) $\ln|x|$;
   (l) $\frac{1}{2}\ln|2x+1|$;
   (m) $x - \ln|1+x|$.
2. (a) $\frac{1}{2}x + \frac{1}{4}\sin(2x) + c$;

   (b) $-\cos x + \frac{1}{3}\cos^3 x + c$ (or equivalents);

   (c) $\int \sin^4 x dx = \frac{3}{8}x - \frac{1}{8}\left(3 + 2\sin^2 x\right)\cos x \sin x + c$.

   (d) Let $F(x) = \frac{3}{8}x - \frac{1}{8}\left(3 + 2\sin^2 x\right)\cos x \sin x$ and let $u = x + \pi/2$. Then, using the fact that $\sin(x+\pi/2) = \cos x$ and $\cos(x+\pi/2) = -\sin x$,
   $$\begin{aligned} F(u) &= \frac{3}{8}u - \frac{1}{8}\left(3 + 2\sin^2 u\right)\cos u \sin u \\ &= \frac{3}{8}x + \frac{3}{8}(\pi/2) - \frac{1}{8}\left(3 + 2\cos^2(x)\right)(-\sin(x)).\cos(x) \\ &= \frac{3}{8}x + \frac{1}{8}\left(3 + 2\cos^2 x\right)\sin x \cos x + \frac{3\pi}{16}. \end{aligned}$$
   But $F(x)$ differentiates to give $\sin^4 x$, so differentiating the above we get
   $$\frac{d}{dx}(F(u)) = \frac{d}{du}(F(u)).\frac{du}{dx} = F'(u).1 = \sin^4(u) = \cos^4 x.$$
   It follows that the expression for $F(u)$ above is an indefinite integral of $\cos^4 x$; in other words, $\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{8}\left(3 + 2\cos^2 x\right)\sin x \cos x + c$.
3. $F(x)$ has stationary points at $x = 0$, $1$, and $2$, which are maximum, inflexion and minimum, respectively. $F(x)$ passes through zero at $x = 0$. Hopefully that's enough info to draw a rough sketch.
4. (a) $\int \frac{1}{x}\,dx$ seems to be $\ln(x) + C$.

   (b) Now $\int \frac{1}{x}\,dx$ seems to be $\ln(-x) + C$.

   (c) The above is best summarised by $\int \frac{1}{x}\,dx = \ln(|x|) + C$, as I'm sure you already know.

   (d) Similarly, we find $\int \frac{g'(x)}{g(x)}\,dx = \ln(|g(x)|) + C$.

For more details, start a thread on the discussion board.