DIFFERENTIATION OF VECTORS

5 minute review. Recap the fact that if a vector $\mathbf{v}$ depends on time, then we can differentiate to get $\frac{d\mathbf{v}}{dt}$ (also written $\dot{\mathbf{v}}$), and that this is calculated componentwise. That is, if $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $\frac{d\mathbf{v}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$. Also cover the facts that

- $\frac{d\mathbf{c}}{dt} = \mathbf{0}$ (the zero vector) if $\mathbf{c}$ is a constant vector;
- $\frac{d}{dt}(m\mathbf{v}) = m\frac{d\mathbf{v}}{dt}$ if m is a scalar;
- $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \cdot \mathbf{v}$;
- $\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{u}}{dt} \times \mathbf{v}$.

Class warm-up. Let $\mathbf{r} = (\cos t, \sin t, 0)$. Show that $\dot{\mathbf{r}}$ is perpendicular to $\mathbf{r}$. Now let $\mathbf{r}$ be any unit vector whose direction depends on t. By writing $\mathbf{r} \cdot \mathbf{r} = 1$ and differentiating, conclude that $\dot{\mathbf{r}}$ is again perpendicular to $\mathbf{r}$ (as long as $\dot{\mathbf{r}} \neq 0$). You could discuss what the implications are if (a) $\mathbf{r}$ represents position, or (b) $\mathbf{r}$ represents velocity.

Problems. Choose from the below.

1. **Standard calculations**. Let $\mathbf{a} = (5t^2, t, -t^3)$ and $\mathbf{b} = (\sin t, -\cos t, 0)$. Find
$$\text{(a) } \frac{d}{dt}(\mathbf{a} \cdot \mathbf{b}); \quad \text{(b) } \frac{d}{dt}(t^2\mathbf{a}); \quad \text{(c) } \frac{d}{dt}(\mathbf{a} \times \mathbf{b}).$$

2. **Circular motion**. An object moves so that its position vector at time t is given by
$$\mathbf{r} = a(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}),$$
where a and ω are constants.
   (a) Show that its velocity is perpendicular to $\mathbf{r}$.
   (b) Show that its acceleration is directed towards the origin and has magnitude proportional to the distance of the object from the origin.

3. **Intersecting paths**. Particle A starts at the point $(2, 2, 6)$ and moves with constant velocity $\mathbf{v}_A = (1, -1, 2)$. Particle B starts at $(1, -1, 10)$ and moves with constant velocity $\mathbf{v}_B = (3, 1, 0)$. Do the paths cross? Do the particles collide?

4. **Particles in magnetic fields**. The equation of motion for a particle of constant mass m and charge e moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is
$$m\frac{d\mathbf{v}}{dt} = e\left(\mathbf{v} \times \mathbf{B}\right).$$
   (a) Find $\mathbf{v} \cdot \frac{d\mathbf{v}}{dt}$.
   (b) By considering the derivative of $\mathbf{v} \cdot \mathbf{v}$, show that the speed of the particle is constant. (Remember that speed is a scalar quantity. This isn't the same as saying that the velocity is constant!)
   (c) Under the assumption that $\mathbf{B}$ is in a fixed direction $\mathbf{B} = B_0\mathbf{k}$, deduce that the component of the velocity in the same direction as $\mathbf{B}$ (that is, the $\mathbf{k}$-direction) is constant.

For the warm-up, $\dot{\mathbf{r}} = (-\sin t, \cos t, 0)$ so $\dot{\mathbf{r}} \cdot \mathbf{r} = -\sin t \cos t + \cos t \sin t = 0$, meaning $\mathbf{r}$ and $\dot{\mathbf{r}}$ are perpendicular. If $\mathbf{r}$ is any unit vector, then $\mathbf{r} \cdot \mathbf{r} = 1$ so $\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} \cdot \mathbf{r} = 0$, giving $2\mathbf{r} \cdot \dot{\mathbf{r}} = 0$ so, again, $\mathbf{r} \cdot \dot{\mathbf{r}} = 0$. As long as $\dot{\mathbf{r}}$ has magnitude greater than zero, this means that $\mathbf{r}$ and $\dot{\mathbf{r}}$ are perpendicular.

Selected answers and hints.

1. (a) $11t \sin t + (5t^2 - 1) \cos t$.

   (b) $(20t^3, 3t^2, -5t^4)$.

   (c) $(t^2(t \sin t - 3 \cos t), -t^2(3 \sin t + t \cos t), (5t^2 - 1) \sin t - 11t \cos t)$.

2. (a) The velocity is $\mathbf{v} = a\omega(-\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j})$. It's easy to show that $\mathbf{v} \cdot \mathbf{r} = 0$ (so that $\mathbf{v}$ and $\mathbf{r}$ are perpendicular).

   (b) The acceleration is $\mathbf{a} = -a\omega^2(\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}) = -\omega^2\mathbf{r}$. This means that $\mathbf{a}$ points in the opposite direction to $\mathbf{r}$, namely towards the origin, and its magnitude is $\omega^2 r$, where r is the distance from the origin.

3. The paths do cross, but the particles do not intersect. Particle A has position vector given by $\mathbf{r}_A = (2, 2, 6) + t(1, -1, 2)$, and particle B has position vector $\mathbf{r}_B = (1, -1, 10) + t(3, 1, 0)$. When $t = 2$, $\mathbf{r}_A = (4, 0, 10)$, which is the same as $\mathbf{r}_B$ at $t = 1$. However, there is no value of t such that $\mathbf{r}_A = \mathbf{r}_B$ simultaneously.

4. (a) Taking the scalar product of both sides with $\mathbf{v}$,

   $$m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = e\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}).$$

   But $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$ because $(\mathbf{v} \times \mathbf{B})$ is perpedicular to $\mathbf{v}$. Hence $m\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 0$, so $\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 0$

   (b) On the one hand, $\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) = \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} = 0$ (from part (a)). On the other hand, $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = v^2$, where v is the speed of the particle. Hence, v^2 is constant, so v is constant.

   (c) Let $B = B_0\mathbf{k}$. Taking scalar products with $\mathbf{k}$ in the given equation,

   $$m\mathbf{k} \cdot \frac{d\mathbf{v}}{dt} = e\mathbf{k} \cdot (\mathbf{v} \times B_0\mathbf{k}) = 0,$$

   since $\mathbf{v} \times B_0\mathbf{k}$ is perpendicular to $\mathbf{k}$. Thus $\mathbf{k} \cdot \frac{d\mathbf{v}}{dt} = 0$. This means that the acceleration in the $\mathbf{k}$-direction is zero, so the $\mathbf{k}$-component of the velocity is constant.

   (Equivalently, $\frac{d}{dt}(\mathbf{k} \cdot \mathbf{v}) = \mathbf{k} \cdot \frac{d\mathbf{v}}{dt} + \frac{d\mathbf{k}}{dt} \cdot \mathbf{v} = \mathbf{k} \cdot \frac{d\mathbf{v}}{dt} = 0$, since $\frac{d\mathbf{k}}{dt} = \mathbf{0}$ as $\mathbf{k}$ is a constant vector. Thus $\mathbf{k} \cdot \mathbf{v}$, which is the $\mathbf{k}$-component of $\mathbf{v}$, must be constant.)

For more details, start a thread on the discussion board.