VECTOR PRODUCTS

5 minute review. Recap the vector product as $\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}| \sin\theta \, \hat{\mathbf{n}}$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ such that $\{\mathbf{u}, \mathbf{v}, \hat{\mathbf{n}}\}$ forms a right-handed system. Also briefly cover determinants of 2×2 and 3×3 matrices, and how to evaluate the vector product of $\mathbf{u} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ as

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & b & c \\ x & y & z \end{vmatrix}.$$

Class warm-up. Given that $\mathbf{a} = (3, -1, -2)$ and $\mathbf{b} = (2, 3, 1)$, find (i) $\mathbf{a} \times \mathbf{b}$, and (ii) $(\mathbf{a} + 2\mathbf{b}) \times (2\mathbf{a} - \mathbf{b})$. Find the latter in two ways (direct calculation/expanding the brackets).

Problems. Choose from the below.

1. **Working with vector products**.

   (a) The vector $\mathbf{a}$ points east with magnitude 2, and $\mathbf{b}$ points north-west with magnitude 3. What is the magnitude and direction of $\mathbf{c} = \mathbf{a} \times \mathbf{b}$? What about $\mathbf{c} \times \mathbf{a}$?

   (b) Let $\mathbf{a}$ and $\mathbf{b}$ be any vectors. What is $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a}$?

   Simplify (i) $(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b})$; (ii) $(\mathbf{a}+3\mathbf{b}) \times (\mathbf{a}-\mathbf{b})$; (iii) $\mathbf{a} \cdot (\mathbf{a} \times (\mathbf{a}+\mathbf{b}))$; (iv) $\mathbf{a} \times (\mathbf{a} \cdot (\mathbf{a} + \mathbf{b}))$, where $\cdot$ is the scalar product and $\times$ is the vector product. (One of these is a trick question.)

2. **Vector product calculations**. Let $\mathbf{a} = (1, 2, 3)$, $\mathbf{b} = (2, 1, 4)$ and $\mathbf{c} = (1, -1, 2)$.

   (a) Calculate $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. What do you notice?

   (b) Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. By using properties of the vector product, guess a corresponding formula for $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and show that it holds in this case. (In fact, these are general formulae which work for any $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.)

3. **Areas and volumes**.

   (a) What is the formula for the area A of a triangle with two sides a and b and an included angle θ? (Hint: draw a diagram, use one of the sides as the base and calculate the perpendicular height.)

   (b) Let PQR be a triangle, and let $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PR} = \mathbf{b}$. By using part (a), express $|\mathbf{a} \times \mathbf{b}|$ in terms of the area A of the triangle. Conclude that $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$, as claimed in the video.

   (c) By using the fact that the volume V of a tetrahedron is given by $V = \frac{1}{3}Ah$, where A is the base area and h is the perpendicular height, show that $V = \frac{1}{6}|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors pointing away from any one of the corners. Why are the modulus signs $| \dots |$ needed here?

For the warm-up, $\mathbf{a}\times\mathbf{b} = 5\mathbf{i} - 7\mathbf{j} + 11\mathbf{k}$ and $(\mathbf{a}+2\mathbf{b})\times(2\mathbf{a}-\mathbf{b}) = -25\mathbf{i} + 35\mathbf{j} - 55\mathbf{k}$.

Selected answers and hints.

1. (a) $\mathbf{c}$ has magnitude $3\sqrt{2}$ and points up; $\mathbf{c}\times\mathbf{a}$ has magnitude $6\sqrt{2}$ pointing north.

   (b) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = 0$, as $\mathbf{a}\times\mathbf{b}$ is perpendicular to $\mathbf{a}$.

   (i) $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = 2\mathbf{b}\times\mathbf{a}$; (ii) $(\mathbf{a}+3\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = 4\mathbf{b}\times\mathbf{a}$; (iii) $\mathbf{a}\cdot(\mathbf{a}\times(\mathbf{a}+\mathbf{b})) = 0$; (iv) $\mathbf{a}\times(\mathbf{a}\cdot(\mathbf{a}+\mathbf{b}))$ cannot be done, because $\mathbf{a}\cdot(\mathbf{a}+\mathbf{b})$ is a scalar not a vector.

2. (a) $(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (1,-13,-7)$, whereas $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (-6,21,-12)$. These are different! (The vector product is not associative, i.e. the order of calculation matters.)

   (b) Since swapping the order in a vector product introduces a minus sign, a good guess is that

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = -\{(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}\} = (\mathbf{c}\cdot\mathbf{a})\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a},$$

   which is true.

3. (a) $A = \frac{1}{2}ab\sin\theta$.

   (b) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta = 2A$. The area of the parallelogram is twice the area of the triangle, as a picture shows.

   (c) The area of the triangle formed by $\mathbf{b}$ and $\mathbf{c}$ is $\frac{1}{2}|\mathbf{b}\times\mathbf{c}|$ as shown above. If the perpendicular height is h, then $h = |\mathbf{a}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and the perpendicular to the base triangle pointing up into the tetrahedron. (Draw a picture at this point!) But $\mathbf{b}\times\mathbf{c}$ is perpendicular to the base triangle, so assuming it points up into the tetrahedron,

$$V = \frac{1}{3}Ah = \frac{1}{6}|\mathbf{a}||\mathbf{b}\times\mathbf{c}|\cos\theta = \frac{1}{6}\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}).$$

   If the $\mathbf{b}\times\mathbf{c}$ points down, out of the tetrahedron, then $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ will be negative, so the modulus is needed.

For more details, start a thread on the discussion board.